13. Morphisms between varieties II

Consider the following very classical problem.

Determine all triples of integers a, b and c such that $a^2 + b^2 = c^2$.

One way to solve this problem is to use some geometry. Consider the circle with equation $x^2 + y^2 = 1$. Pick a point of this conic, say $(0, 1)$ in the ordinary plane. Consider picking a line l passing through p. This line will intersect the circle at one further point, say q. Let us find the coordinates of q.

Now the general line through $(0, 1)$ is of the form

$$y - 1 = -tx,$$

for some $t \in K$, at least if the line is not vertical. Substituting into

$$x^2 + y^2 = 1,$$

we get

$$x^2 + (-tx + 1)^2 = 1,$$

so that

$$(t^2 + 1)x^2 - 2tx = 0.$$

Thus either $x = 0$, the solution we already have, or

$$(t^2 + 1)x - 2t = 0,$$

so that

$$x = \frac{2t}{t^2 + 1}.$$

In this case

$$y = -tx + 1 = \frac{1 - t^2}{t^2 + 1}.$$

In this way, we get a morphism

$$f \colon \mathbb{A}^1 - \{\pm i\} \longrightarrow \mathbb{A}^2,$$

where

$$t \longrightarrow \left(\frac{2t}{t^2 + 1}, \frac{1 - t^2}{t^2 + 1} \right)$$

Note that we can reverse this process. That is, we can start with a point (x, y) of C and obtain a point of the x-axis, simply by projection. In fact this map is defined for any point away from the line $y = 1$. If we have a point (x, y) then we send this to a point z, where the three points $(0, 1)$, (x, y) and $(z, 0)$ are collinear. Now the reciprocal of the slope of the line connecting $(0, 1)$ to $(z, 0)$ is $-z$, so that

$$\frac{x}{y - 1} = -z.$$

Thus the map is

$$\mathbb{A}^2 - \{y = 1\} \longrightarrow \mathbb{A}^1,$$
$$(x, y) \longrightarrow \frac{x}{1-y}.$$

When we restrict to C we get a morphism outside of $(0, 1)$. What happens when we projectivise?

In this case, $x = X/Z$, $y = Y/Z$ and, at least symbolically we get

$$[X : Y : Z] \longrightarrow [1 : \frac{X/Z}{1 - Y/Z}] = [1 : \frac{X}{Z - Y}].$$

Now note that, at least outside the locus $Z = Y$,

$$[1 : \frac{X}{Z - Y}] = [Z - Y : X].$$

So that it makes sense to extend the map by the rule

$$[X : Y : Z] \longrightarrow [Z - Y : X].$$

This gives us a well-defined map, except on the locus $Z = Y$, $X = 0$, that is the point $[0 : 1 : 1]$. The key point to observe is that even though there is no way to extend this morphism, which is defined on $\mathbb{P}^2 - [0 : 1 : 1]$, to the whole of $\mathbb{P}^2$ (geometrically this is clear, since we would somehow be picking out one line from amongst all lines through $[0 : 1 : 1]$, and the natural symmetry says we cannot do this), in fact it does make sense to extend this map to the whole of $C = V(X^2 + Y^2 - Z^2)$. Again geometrically this is clear. In the limit as y tends to one, the line tends to a horizontal line, a line of zero slope. In fact this is equally clear algebraically. On the curve $X^2 + Y^2 = Z^2$, so that

$$(Z - Y)(Z + Y) = X^2.$$

Given this, on the locus where $Z + Y \neq 0$ and $X \neq 0$,

$$\begin{aligned}
[Z - Y : X] &= [(Z + Y)(Z - Y) : (Z + Y)X] \\
&= [X^2 : (Y + Z)X] \\
&= [X : Y + Z].
\end{aligned}$$

Thus we could have defined the morphism equally well by using $[X : Y + Z]$. Note this gives us what we expect geometrically; the point $[0 : 1 : 1]$ would be sent to the point $[0 : 2] = [0 : 1]$, using the last prescription.

In other words, a priori, the map

$$C - \{[0 : 1 : 1]\} \longrightarrow \mathbb{P}^2,$$

given as

$$[X : Y : Z] \longrightarrow [Z - Y : X],$$

would not seem to extend to $[0 : 1 : 1]$. In fact, using our working definition, it does not. The point is, then, that our working definition is not quite right. Let us think carefully through how we want to change our definition, so as to add this map as a morphism. We are given $X \subset \mathbb{P}^n$ and a morphism defined on two "large" subsets of X, a la working definition. That is we are given $U \subset X$, and $F_0, F_1, \dots, F_m$ without common zeroes on U, similarly $V \subset X$, with $G_0, G_1, \dots, G_m$ without common zeroes on V. On the intersection, we have

$$[F_0 : F_1 : \dots : F_m] = [G_0 : G_1 : \dots : G_m]$$

so that they represent the same function on the intersection $U \cap V$. We would then like to extend the morphism to the whole of $U \cup V$, definining the map piecewise.

Definition 13.1. *A* ***quasi-projective variety*** *is a locally closed subset of $\mathbb{P}^n$.*

Example 13.2. *Every affine variety $V \subset \mathbb{A}^n$ is a quasi-projective variety. Indeed V is closed subset of $\mathbb{A}^n$ so that it is the intersection of the closure W of V with the open set $U_0 \simeq \mathbb{A}^n$. Similarly, every quasi-affine variety is a quasi-projective variety.*

Conversely, note that if $V \subset \mathbb{P}^n$ is a quasi-projective variety then $V_i = V \cap U_i \subset U_i \simeq \mathbb{A}^n$ is a quasi-affine variety.

Definition 13.3. *Let*

$$f : V \longrightarrow W,$$

be a map between two quasi-projective varieties $V \subset \mathbb{P}^m$ and $W \subset \mathbb{P}^n$. Let V_α and W_i be the quasi-affine covers defined above and let

$$U_{\alpha,i} = f^{-1}(W_i) \cap V_\alpha.$$

We say that f is a morphism, if the resriction

$$f|_{W_{i,\alpha}} : W_{\alpha,i} \longrightarrow U_i$$

is a morphism of quasi-affine varieties.

Note that the coordinates of $f|_{W_{i,\alpha}}$ are regular functions on a quasi-affine variety.

This gives us a category, the category of quasi-projective varieties and morphisms.

We now relate this definition, to our previous working definition. For example, suppose we are given a map between projective varieties

$$f : X \longrightarrow Y,$$

$X \subset \mathbb{P}^m$ and $Y \subset \mathbb{P}^n$, which is given by a collection of homogeneous polynomials $F_0, F_1, \dots, F_n$ of the same degree d, which don't vanish simultaneously,

$$[X_0 : X_1 : \dots : X_m] \longrightarrow [F_0 : F_1 : \dots : F_n].$$

On the open subsets $X_\alpha \neq 0$ and $Y_i \neq 0$ this reduces to the map

$$(x_0, x_1, \dots, \hat{x_\alpha}, \dots, x_n) \longrightarrow (f_0, f_1, \dots, \hat{f_i}, \dots, f_n),$$

where

$$x_j = \frac{X_j}{X_\alpha} \qquad \text{and} \qquad f_j = \frac{Y_j}{Y_i}.$$

We note one very curious

Proposition 13.4. *Let $X \subset \mathbb{P}^n$ be a projective variety.*

Then we may find an embedding of X into some $\mathbb{P}^N$ such that X is defined by linear polynomials and quadratic polynomials of rank at most 4.

Lemma 13.5. *The d-uple embedding*

$$\mathbb{P}^n \longrightarrow \mathbb{P}^N,$$

is an isomorphism and a homeomorphism with its image Y, which is a closed subset of $\mathbb{P}^N$ with equations

$$Z_I Z_J = Z_{I'} Z_{J'} \qquad \textit{for all labels} \qquad I + J = I' + J'$$

Proof. The d-uple embedding is certainly a morphism. We already showed that $Y \subset \mathbb{P}^N$ is closed. One can check that Y is cut out by the given equations.

We now try to write down the inverse map. The image Y is contained in the open affine cover $V_i = V_I$, where I ranges over the pure powers, $I = (0, 0, \dots, 0, d, 0, \dots 0)$ (that is, $X^I = X_i^d$). The inverse image is then the open affine U_i, $X_i \neq 0$, and we get morphisms

$$f_i \colon U_i \longrightarrow V_i.$$

We write down do the inverse image in the case of f_0, so that we want to define a map $g_0 \colon V_0 \longrightarrow U_0$ a morphism, which will turn out to be the inverse of f_0. In this case, we send

$$[Z_I] \longrightarrow [Z_{(d,0,0,\dots,0)} : Z_{(d-1,1,0,\dots,0)} : Z_{(d-1,0,1,\dots,0)} : \dots : Z_{(d-1,0,0,\dots,1)}].$$

This map is certainly a morphism. Moreover it is not hard to see that it is the inverse of f_0. By symmetry this gives us maps g_i, inverses of f_i. Since the inverse map $g = f^{-1}$ is unique, provided it exists, compatibility on overlaps is guaranteed. $\square$

Lemma 13.6. *Let X be a projective variety and let F be a homogeneous polynomial of degree d. Then the set*

$$U_F = \{ x \in X \mid F(x) \neq 0 \},$$

is an open affine subset of X.

Proof. Let Y be the image in $\mathbb{P}^N$ of X under the d-uple embedding. By (13.5) Y is isomorphic to X. Suppose that

$$F(X) = \sum_I a_I X^I.$$

Then $F = 0$ corresponds to the locus

$$L = \sum a_I Z_I = 0,$$

which is a hyperplane in $\mathbb{P}^N$. Since the complement of any hyperplane is a copy of affine space (just change coordinates so that the hyperplane is given as $Z_N = 0$) the corresponding subset of Y, U_L, is an affine subset. As U_L is isomorphic to U_F, the result follows. $\square$

Proof of (13.4). Suppose that X is defined by $F_1, F_2, \dots, F_k$. Note that we may assume that $F_1, F_2, \dots, F_k$ all have the same degree. Indeed the vanishing locus of $X_0, X_1, \dots, X_n$ is empty so that the vanishing locus of F and X_0F, X_1F, $\dots$, X_nF coincide. So just top up the degrees until they are all the same.

Now consider the d-uple embedding ν_d. The image of X inside $\nu_d(\mathbb{P}^n)$ is defined by a collection of linear polynomials; if $F_i = \sum a_I X^I$ then we need the linear polynomial $\sum a_I Z_I$. But we already know that $\nu_d(\mathbb{P}^n)$ is defined by quadratic polynomials of the form $Z_I Z_J = Z_{I'} Z_{J'}$, which has rank at most four. $\square$