15. Products and fibre products

Definition 15.1. *Let*

$$\mathbb{P}^m \times \mathbb{P}^n \longrightarrow \mathbb{P}^{mn+m+n},$$

denote the map given by

$$([X_0, X_1, \dots, X_m], [Y_0, Y_1, \dots, Y_n]) \longrightarrow [X_i Y_j].$$

This map is easily seen to be a bijection and the image is a closed subset, defined by the quadratic polynomials

$$Z_{ij} Z_{kl} = Z_{il} Z_{kj},$$

(where of course Z_{ij} corresponds to $X_i Y_j$). The image V is called the **Segre variety**, and we define the product using this map, that is, we are aiming for:

Proposition 15.2. *Let $X \subset \mathbb{P}^m$ and $Y \subset \mathbb{P}^n$. Then the image of $X \times Y$ under the Segre map is the product (in the sense of category theory) of X and Y.*

Lemma 15.3. *The Segre Variety V is the product, in the sense of category theory, of $\mathbb{P}^m$ and $\mathbb{P}^n$.*

Proof. We have to exhibit two morphisms $p\colon V \longrightarrow \mathbb{P}^m$ and $q\colon V \longrightarrow \mathbb{P}^n$ and show that they satisfy the universal property. Fix l and let $U_l \subset V$ be the open subset where at least one of Z_{il} is non-zero. Define a map

$$U_l \longrightarrow \mathbb{P}^m,$$

by sending $[Z_{ij}]$ to $[Z_{il}]$. This is clearly a morphism, and these maps agree on overlaps. Moreover, varying l, the U_l cover V so that we get a morphism on the whole of V.

By symmetry, this gives us two morphisms p and q. Moreover, under the identification of V with $\mathbb{P}^m \times \mathbb{P}^n$, it is clear that p and q are the ordinary projection maps. Since $\mathbb{P}^m \times \mathbb{P}^n$ is a product in the category of sets, given any morphisms $p'\colon Z \longrightarrow \mathbb{P}^m$ and $q'\colon Z \longrightarrow \mathbb{P}^n$, there is an induced unique function

$$f\colon Z \longrightarrow V.$$

It suffices to check that f is a morphism. We check this locally. Let $U_{ij} \subset V$ be the locus where $Z_{ij} \neq 0$. Then U_{ij} corresponds to $U_i \times U_j$. We first check that U_{ij} is isomorphic to $\mathbb{A}^{m+n}$. By symmetry, we may assume that $i = j = 0$. In this case, dehomogenising, the equations of $U = U_{00}$ become

$$z_{ij} = z_{i0} z_{0j} \qquad \text{and} \qquad z_{ij} z_{kl} = z_{il} z_{kj}.$$

Define a morphism

$$\mathbb{A}^{n+m} \longrightarrow U,$$

by the rule

$$(z_{10}, z_{20}, \dots, z_{m0}, z_{01}, z_{02}, \dots z_{0n}) \longrightarrow (z_{i0}z_{0j}).$$

This is clearly a morphism with image U and it is not hard to show that projection on the first $m+n$ factors is the inverse. Thus $U_{ij} \simeq \mathbb{A}^{n+m}$.

It is easy to check that $\mathbb{A}^{m+n}$ is the product of $\mathbb{A}^m$ and $\mathbb{A}^n$. Thus, by the universal property of the product, f_{ij}, the restriction of f to the inverse image of U_{ij}, is a morphism. $\square$

The general case, follows by the same argument, provided we can prove that the image of $X \times Y$ is a closed subset. In other words we have to say something about which subsets of V are closed.

Definition 15.4. *Let $F(X,Y)$ be a polynomial in $X_0, X_1, \dots, X_m$ and $Y_0, Y_1, \dots, Y_n$. We say that $F(X,Y)$ is* ***bihomogeneous*** *of bi-degree (d,e) if it is homogeneous of degree d in the variables $X_0, X_1, \dots, X_m$ and of degree e in the variables $Y_0, Y_1, \dots, Y_n$.*

For example, $X_0Y_1^2 + X_1Y_0Y_1$ is bihomogeneous of bi-degree $(1,2)$.

Note that the zero locus of a bihomogeneous polynomial is a well-defined subset of the product.

Lemma 15.5. *Let $Z \subset V$ be a subset defined by bihomogeneous polynomials.*

Then Z is a closed subset.

Proof. Topping up the degree, we may as well assume that X is defined by bihomogeneous polynomials F of bi-degree (d,d). It suffices then to prove that there is a polynomial G on $\mathbb{P}^{mn+m+n}$ which pulls back to F. By linearity, it suffices to prove this for monomials. But Z_{ij} pulls back to X_iY_j, and we can clearly build any monomial X^IY^J, as a product of such monomials, provided that X^I and Y^J have the same degree. $\square$

Proof of (15.2). By (15.5) the image of $X \times Y$ is a closed subset of the Segre variety, under the Segre map, and the rest of the the proof goes through as before. $\square$

It is interesting to see what happens in a specific example. Suppose we take the twisted cubic in $\mathbb{P}^3$. This lies in the quadric $XW = YZ$, that is, it lies in the Segre variety. Now it also lies in the quadric $Y^2 - XZ$. Pulling back to $\mathbb{P}^1 \times \mathbb{P}^1$ we get the bihomogeneous polynomial

$$(X_0Y_1)^2 - (X_0Y_0)(X_1Y_0) = X_0(X_0Y_1^2 - X_1Y_0^2).$$

Now the equation $X_0 = 0$ corresponds to a line in the quadric (see below), and what is left defines the twisted cubic. Thus the twisted cubic is defined by a bi-homogeneous polynomial of type $(1, 2)$.

It is also interesting to see what happens to $\{p\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{q\}$. Indeed $[\lambda_0 : \lambda_1] \times [Y_0 : Y_1]$ is sent to $[\lambda_0 Y_0 : \lambda_0 Y_1 : \lambda_1 Y_0 : \lambda_1 Y_1]$, which is the parametric form of a line. In fact the equations of the line are $\lambda_1 X = \lambda_0 Z$ and $\lambda_1 Y = \lambda_0 W$.

Similarly $[X_0 : X_1] \times [\mu_0 : \mu_1]$ is sent to the line $[\mu_0 X_0 : \mu_1 X_0 : \mu_0 X_1 : \mu_1 X_1]$. This line has equations $\mu_1 X = \mu_0 Y$ and $\mu_1 Z = \mu_0 W$.

It follows that the Segre variety $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ is covered by two 1-parameter families of lines.

There is in fact another way to look at all of this. Let V and W be two vector spaces of dimension two. Consider the natural map

$$V \times W \longrightarrow V \otimes W.$$

This induces a map

$$\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3.$$

Let us calculate what this map is in terms of coordinates. A general vector $v \in V$ has the form $v = ae + bf$, where $\{e, f\}$ is a basis of V. Similarly a general vector $w \in W$ is of the form $cg + dh$, where $\{g, h\}$ is a basis of W. Thus the pair (v, w) is sent to

$$\begin{aligned} v \otimes w &= (ae + bf) \otimes (cg + dh) \\ &= ac(e \otimes g) + ad(e \otimes h) + bc(f \otimes g) + bd(f \otimes h). \end{aligned}$$

The induced map is then

$$\begin{aligned} ([v], [w]) &\longrightarrow [ac(e \otimes g) + ad(e \otimes h) + bc(f \otimes g) + bd(f \otimes h)] \\ ([a : b], [c : d]) &\longrightarrow [v \otimes w] = [ac : ad : bc : bd], \end{aligned}$$

which is clearly the Segre map. Thus the Segre variety consists of all tensors of rank one. The two families of lines, are given as $[v] \times \mathbb{P}^1$ and $\mathbb{P}^1 \times [w]$.

Clearly this generalises in an obvious way to the general Segre variety, which is covered by two families of linear spaces. A family of linear spaces of dimension m, parametrised by $\mathbb{P}^n$ and a family of linear spaces of dimension n, parametrised by $\mathbb{P}^m$.

This also sheds some light on the fact that the twisted cubic is not the intersection of two quadrics. First a quick digression on quadrics, the zero locus of a quadratic polynomial. A quadratic polynomial in the variables $X_0, X_1, \dots, X_n$ is the same as an element of $\operatorname{Sym}^2(V^*)$. These can be identified with symmetric bilinear forms. As such any quadratic polynomial (whence a quadric) has a rank. Since the only invariant of a symmetric bilinear form over an algebraically closed field is its rank, quadrics are classified up to projective equivalence by their rank.

It is interesting to see how this works in practice. If $Q \subset \mathbb{P}^n$ is a quadric of rank $r \leq n+1$ then we can always choose coordinates so that Q is given as

$$X_0^2 + X_1^2 + \cdots + X_{r-1}^2 = 0.$$

(Actually not quite; in characteristic two one must look at $X_0X_1 + X_1X_2 + \dots$ with a square at the end depending on the parity of r). Now if any variety X is defined by homogeneous polynomials which don't involve the last variable then X contains the point $q = [0 : 0 : \cdots : 1]$. In fact if $p \in X$ then the line $\langle p, q \rangle$ is contained in X. More generally, if the equations defining X don't involve the last $n - r$ variables, then X contains the linear space Λ given by $X_0 = X_1 = \dots X_{r-1} = 0$ and if $p \in X$ then so is the linear space $\langle p, \Lambda \rangle$. In this case we say that X is a **cone** over Λ and we call Λ the **vertex**. Note that to specify X, look at the variety Y you get by considering the zero locus of the polynomials in $\mathbb{P}^{r-1}$ and then joining every point of Y to every point of Λ.

Back to quadrics. In $\mathbb{P}^1$ there are two possibilities. If the rank is 1 then we must have X^2 and we get one point. If the rank is 2 we get $X^2 + Y^2$ (or XY in characteristic two) and we get two points. So there are three possibilities for a conic in $\mathbb{P}^2$. If the rank is at most two, we get the cone over one or two points, that is a line (counted twice, as it were) or a pair of lines. Otherwise we get a smooth conic. In $\mathbb{P}^3$ there are four possibilties. A double plane, a pair of planes, a cone (in the classic sense) or the Segre variety.

Back to twisted cubics. If one of the quadrics has maximal rank 4 (or better one of the quadrics in the pencil, which is in fact always true), then it is projectively equivalent to the Segre variety. In this case the other quadric cuts out a curve of bi-degree $(2, 2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. As the twisted cubic has bi-degree $(1, 2)$, it follows that we get not only the twisted cubic, but a line (something of bi-degree $(1, 0)$), so that the union has bi-degree $(2, 2)$. Now the line is a fibre of one of the rulings, and a general fibre meets the cubic in two points (since a quadratic polynomial has two roots in general).

In fact projecting a curve C of bi-degree (d, e) to either factor defines a morphism $C \longrightarrow \mathbb{P}^1$ which has degree d (respectively e), that is, the typical fibre contains d points (at least in characteristic zero).

Theorem 15.6. *Let $\phi\colon X \longrightarrow B$ and $\psi\colon Y \longrightarrow B$ be two morphisms of quasi-projective varieties.*

Then the set

$$I = \{\, (x,y,b) \in X \times Y \mid b = \phi(x) = \psi(y) \,\} \subset X \times Y \times B,$$

is the fibre product of ϕ and ψ in the category of varieties.

Proof. We have already seen that the two projection maps $p\colon X \times Y \times B \longrightarrow X$ and $q\colon X \times Y \times B \longrightarrow Y$ are morphisms. Suppose we are given two morphisms $f\colon Z \longrightarrow X$ and $g\colon Z \longrightarrow Y$ such that $\phi \circ f = \psi \circ g$. By composition we get two morphisms to B and by the universal property of the products $X \times B$ and $Y \times B$, we get two morphisms $f'\colon Z \longrightarrow X \times B$ and $g'\colon Z \longrightarrow Y \times B$. By the universal property of the product $X \times B \times Y \times B$, there is an induced morphism $Z \longrightarrow X \times Y \times B \times B$. Note that under projection to the last two factors, the image always lies in the diagonal

$$\{\, (b,b) \mid b \in B \,\},$$

which is easily to be a copy of B. So we get a morphism $Z \longrightarrow X \times Y \times B$. By the universal property of the fibre product, the image lands in I.

So the only thing to check is that I is a quasi-projective variety. □

Definition-Lemma 15.7. *Let $\phi\colon X \longrightarrow B$ be a morphism of quasi-projective varieties. The **graph** of ϕ is the closed set*

$$\Gamma_\phi = \{\, (x,b) \mid \phi(x) = b \,\} \subset X \times B.$$

It is isomorphic to X via the first projection map.

Proof. The only things to check are that Γ_ϕ is closed and the first projection map is an isomorphism. Since we can check this locally, we may assume that X and B are affine. We may then assume that $B = \mathbb{A}^n$ and that ϕ is given as

$$(x_1, x_2, \dots, x_m) \longrightarrow (f_1/g, f_2/g, ..., f_n/g),$$

where g does not vanish on X. In this case the graph is given by the equations

$$gy_i = f_i,$$

where $y_1, y_2, \dots, y_n$ are coordinates on $\mathbb{A}^n$. To see that the first map is an isomorphism, one can use the fact that the graph is in fact the fibre product of the identity $X \longrightarrow X$ and the morphism $\phi\colon X \longrightarrow B$ over B. The inverse map of the first projection is

$$(x_1, x_2, \dots, x_m) \longrightarrow (x_1, x_2, \dots, x_m, f_1/g, f_2/g, ..., f_n/g). \qquad \square$$

Lemma 15.8. *$I \subset X \times Y \times B$ is closed.*

Proof. This is easy. The graphs of ϕ and ψ define two closed subsets of $X \times Y \times B$,

$$\{\,(x,y,b)\,|\,\phi(x) = b\,\} \qquad \text{and} \qquad \{\,(x,y,b)\,|\,\psi(y) = b\,\},$$

and I is the intersection. $\square$