17. First properties of schemes

We start with some basic properties of schemes.

Definition 17.1. *We say that a scheme is* ***connected*** *(respectively* ***irreducible****) if its topological space is connected (respectively irreducible).*

Definition 17.2. *We say that a scheme is* ***reduced*** *if $\mathcal{O}_X(U)$ contains no nilpotent elements, for every open set U.*

Remark 17.3. *It is straightforward to prove that a scheme is reduced if and only if the stalk of the structure sheaf at every point contains no nilpotent elements.*

Definition 17.4. *We say that a scheme X is* ***integral*** *if for every open set $U \subset X$, $\mathcal{O}_X(U)$ is an integral domain.*

Proposition 17.5. *A scheme X is integral if and only if it is irreducible and reduced.*

Proof. Suppose that X is integral. Then X is surely reduced. Suppose that X is reducible. Then we can find two non-empty disjoint open sets U and V. But then

$$\mathcal{O}_X(U \cup V) \simeq \mathcal{O}_X(U) \times \mathcal{O}_X(V),$$

which is surely not an integral domain.

Now suppose that X is reduced and irreducible. Let $U \subset X$ be an open set and suppose that we have f and $g \in \mathcal{O}_X(U)$ such that $fg = 0$. Set

$$Y = \{\, x \in U \mid f_x \in m_x \,\} \qquad \text{and} \qquad Z = \{\, x \in U \mid g_x \in m_x \,\}.$$

Then Y and Z are both closed and by assumption $Y \cup Z = U$. As X is irreducible, one of Y and Z is the whole of U, say Y. We may assume that $U = \operatorname{Spec} A$ is affine. But then $f \in A$ belongs to the intersection of all the prime ideals of A, which is the zero ideal, as A contains no nilpotent elements. □

Definition 17.6. *We say that a scheme X is* ***locally Noetherian****, if there is an open affine cover, such that the corresponding rings are Noetherian. If in addition the topological space is compact, then we say that X is* ***Noetherian****.*

Remark 17.7. *There are examples of schemes whose topological space is Noetherian which are not Noetherian schemes.*

A key issue in this definition is whether or not we can replace an open cover, by every affine cover.

Proposition 17.8. *A scheme X is locally Noetherian if and only if for every open affine $U = \operatorname{Spec} A$, A is a Noetherian ring.*

Proof. It suffices to prove that if X is locally Noetherian, and $U = \operatorname{Spec} A$ is an open affine subset then A is a Noetherian ring.

We first show that U is locally Noetherian. Suppose that $V = \operatorname{Spec} B$ is an open affine on X where B is a Noetherian ring. Then $U \cap V$ can be covered by open sets of the form $V_f = \operatorname{Spec} B_f$, where $f \in B$. As B is a Noetherian ring then so is B_f. As open sets of the form V cover X, U is covered by open affines, which are the spectra of Noetherian rings. So U is locally Noetherian.

Replacing X by U, we are reduced to proving that if $X = \operatorname{Spec} A$ is locally Noetherian then A is Noetherian. Let $V = \operatorname{Spec} B$, be an open subset of X, where B is a Noetherian ring. Then there is an element $f \in A$ such that $U_f \subset V$. Let g be the image of f in B. As

$$X \supset U_f = U_g \subset V,$$

we have an isomorphism of rings $A_f \simeq B_g$, so that A_f is Noetherian. So we can cover X by open subsets $U_f = \operatorname{Spec} A_f$, with A_f Noetherian. As X is compact, we may assume that we have a finite cover. Now apply (17.9). □

Lemma 17.9. *Let A be a ring, and let $f_1, f_2, \dots, f_r$ be elements of A which generate the unit ideal.*

If A_{f_i} is Noetherian, for $1 \leq i \leq r$ then so is A.

Proof. Suppose that we have an ascending chain of ideals,

$$\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \mathfrak{a}_3 \subset \dots,$$

of A. Then for each i,

$$\phi_i(\mathfrak{a}_1) \cdot A_{f_i} \subset \phi_i(\mathfrak{a}_2) \cdot A_{f_i} \subset \phi_i(\mathfrak{a}_3) \cdot A_{f_i} \subset \dots,$$

is an ascending chain of ideals inside A_{f_i}, where $\phi_i \colon A \longrightarrow A_{f_i}$ is the natural map. As each A_{f_i} is Noetherian, all of these chains stabilise. But then the first chain stabilises, by (17.10). □

Lemma 17.10. *Let A be a ring, and let $f_1, f_2, \dots, f_r$ be elements of A which generate the unit ideal. Suppose that $\mathfrak{a}$ is an ideal and let $\phi_i \colon A \longrightarrow A_{f_i}$ be the natural maps. Then*

$$\mathfrak{a} = \bigcap_{i=1}^{r} \phi_i^{-1}(\phi_i(\mathfrak{a}) \cdot A_{f_i}).$$

Proof. The fact that the LHS is included in the RHS is clear. Conversely suppose that b is an element of the RHS. In this case

$$\phi_i(b) = \frac{a_i}{f^{n_i}},$$

for some $a_i \in \mathfrak{a}$ and some positive integer n_i. As there are only finitely many indices, we may assume that $n = n_i$ is fixed. But then

$$f^{m_i}(f^n b - a_i) = 0,$$

for $1 \leq i \leq r$. Once again, we may assume that $m = m_i$ is fixed. It follows that $f_i^N b \in \mathfrak{a}$, for $1 \leq i \leq r$, where $N = n + m$. Let I be the ideal generated by the Nth powers of $f_1, f_2, \dots, f_r$. As the radical of I contains 1, I contains 1. Hence we may write

$$1 = \sum_i c_i f_i^N.$$

But then

$$b = \sum_i c_i f_i^N b \in \mathfrak{a}.$$

□

Definition 17.11. *A morphism $f\colon X \longrightarrow Y$ is* ***locally of finite type*** *if there is an open affine cover $V_i = \operatorname{Spec} B_i$ of Y, such that $f^{-1}(V_i)$ is a union of affine sets $U_{ij} = \operatorname{Spec} A_{ij}$, where each A_{ij} is a finitely generated B_i-algebra. If in addition, we can take U_{ij} to be a finite cover of $f^{-1}(V_i)$, then we say that f is of* ***finite type****.*

Definition 17.12. *We say that a morphism $f\colon X \longrightarrow Y$ is* ***finite*** *if we may cover Y by open affines $V_i = \operatorname{Spec} B_i$, such that $f^{-1}(V_i) = \operatorname{Spec} A_i$ is an affine set, where A_i is a finitely generated B_i-module.*

In both cases, it is straightforward to prove that we can take V_i to be any affine subset of Y.

Example 17.13. *Let*

$$f\colon \mathbb{A}^1_k - \{0\} \longrightarrow \mathbb{A}^1_k,$$

by the natural map given by the natural localisation map

$$k[x] \longrightarrow k[x]_x.$$

As an algebra over $k[x]$, the ring $k[x]_x \simeq k[x, x^{-1}]$ is generated by x^{-1}, so that f is of finite type. However the $k[x]$-module $k[x, x^{-1}]$ is not finitely generated (there is no way to generate all the negative powers of x), so that f is not finite.

Definition 17.14. *Let X be a scheme and let U be an open subset of X. Then the pair $(U, \mathcal{O}_U = \mathcal{O}_X|_U)$ is a scheme, which is called an **open subscheme** of X. An **open immersion** is a morphism $f\colon X \longrightarrow Y$ which induces an isomorphism of X with an open subset of Y.*

Definition 17.15. *A **closed immersion** is a morphism of schemes $\phi = (f, f^{\#})\colon Y \longrightarrow X$ such that f induces a homeomorphism of Y with a closed subset of X and futhermore the map $f^{\#}\colon \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ is surjective. A **closed subscheme** of a scheme X is an equivalence class of closed immersions, where we say that two closed immersions $f\colon Y \longrightarrow X$ and $f'\colon Y' \longrightarrow X$ are equivalent if there is an isomorphism $i\colon Y' \longrightarrow Y$ such that $f' = f \circ i$.*

Despite the seemingly tricky nature of the definition of a closed immersion, in fact it is easy to give examples of closed subschemes of an affine variety.

Lemma 17.16. *Let A be a ring and let $\mathfrak{a}$ be an ideal of A. Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} A/\mathfrak{a}$.*

Then Y is a closed subscheme of X.

Proof. The quotient map map $A \longrightarrow A/\mathfrak{a}$ certainly induces a morphism of schemes $\phi\colon Y \longrightarrow X$. f is certainly a homeomorphism of Y with $V(\mathfrak{a})$ and $f^{\#}\colon \mathcal{O}_X \longrightarrow f_*\mathcal{O}_Y$ is surjective as the map on stalks is induced by the quotient map, which is surjective. □

In fact, it turns out that every closed subscheme of an affine scheme is of this form. It is interesting to look at some examples.

Example 17.17. *Let $X = \mathbb{A}^2_k$. First consider $\mathfrak{a} = \langle y^2 \rangle$. The support of Y is the x-axis. However the scheme Y is not reduced, even though it is irreducible.*

It is clear from this example that in general there are many closed subschemes with the same support (equivalently there are many ideals with the same radical). For example,

Example 17.18. *consider the ideals $\langle x^2, xy, y^2 \rangle$, the double of the maximal ideal of a point and the ideal $\langle x, y^2 \rangle$. They both defines subschemes of $\mathbb{A}^2_k$ supported at the origin.*

Finally consider:

Example 17.19. *$\langle x^2, xy \rangle$. The support of this ideal is the y-axis. But this time the only local ring which has nilpotents is the local ring of the origin.*

We call the origin an **embedded point.**

Definition 17.20. *Let V be an irreducible affine variety with coordinate ring A and let W be a closed irreducible subvariety, defined by the prime ideal $\mathfrak{p}$. Then we can associate two affines schemes $Y \subset X$ to $W \subset V$. Let $X = \operatorname{Spec} A$ and define Y by $\mathfrak{p}$. The n**th infinitessimal neighbourhood of** Y **in** X, denoted Y_n, is the closed subscheme of X corresponding to $\mathfrak{p}^n$.*

Note that the nth infinitessimal neighbourhood of Y in X is a closed subscheme whose support coincides with Y, but whose structure sheaf contains lots of nilpotent elements. As the name might suggest, Y_n carries more information about how Y sits inside X, than does Y itself.

Note that if a scheme X has a topological space with one point, then X must be affine, and the stalk of the structure sheaf at the unique point completely determines X, and this ring has exactly one prime ideal. Moreover a morphism of X into another scheme Y, is equivalent to picking a point y of Y and a morphism of local rings

$$\mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}.$$

But to give a morphism of local rings is the same as to give an inclusion of the quotients of the maximal ideals. Thus to give a morphism of $X = \{x\}$ into Y, sending x to y, we need to specify an inclusion of the residue field of x into the residue field of y.