5. Coordinate rings

Recall the following version of the Nullstellensatz:

Theorem 5.1 (Weak Nullstellensatz)**.** *Let K be an algebraically closed field.*

Then an ideal $\mathfrak{m} \triangleleft R = K[x_1, x_2, \dots, x_n]$ is maximal if and only if it has the form

$$\mathfrak{m}_p = \langle x_1 - a_1, x_2 - a_2, \dots, x_n - a_n \rangle,$$

for some point $p = (a_1, a_2, \dots, a_n) \in K^n$.

Note that with this formulation it is clear why we need K to be algebraically closed. Indeed $I = \langle x^2 + 1 \rangle$ over $\mathbb{R}$ is in fact maximal and the vanishing locus is empty.

Another way to restate the Nullstellensatz is to observe that it establishes an inclusion reversing correspondence between ideals and closed subsets of $\mathbb{A}^n$. However this is just the tip of the iceberg.

Definition 5.2. *Let $X \subset \mathbb{A}^n$ be a closed subset.*

The ***coordinate ring of*** *X, denoted $A(X)$, is the quotient*

$$K[X]/I(X).$$

Corollary 5.3. *Let $X \subset \mathbb{A}^n$ be an affine subvariety.*

There is a correspondence between the points of X and the maximal ideals of the coordinate ring $A(X)$.

Proof. Recall that there is a correspondence between ideals in $R = K[x_1, x_2, \dots, x_n]$ containing I and ideals in the quotient R/I. So there is a correspondence between maximal ideals of R/I and maximal ideals of R containing I.

But an ideal

$$\mathfrak{m}_p = \langle x_1 - a_1, x_2 - a_2, \dots, x_n - a_n \rangle,$$

contains I if and only if $p \in X$ and so we are done by (5.1). □

In fact this correspondence is natural. To prove this, we have to reinterpret the coordinate ring.

Proposition 5.4. *If $X \subset \mathbb{A}^n$ is an affine variety then the ring of regular functions $\mathcal{O}_X(X)$ is isomorphic to the coordinate ring.*

Proof. Let $\pi\colon K[X] \longrightarrow \mathcal{O}_X(X)$ be the map which sends a polynomial f to the obvious regular function ϕ, $\phi(x) = f(x)$. It is clear that π is a ring homomorphism, with kernel $I(X)$. It suffices, then, to prove that π is surjective.

Let ϕ be a regular function on X. By definition there is an open cover U_i of X and rational functions f_i/g_i such that ϕ is locally given by f_i/g_i. As X is Noetherian, we may assume that each U_i is irreducible. We may assume that $U_i = U_{h_i}$ for some regular function h_i, as such subsets form a base for the topology. Replacing f_i by $f_i h_i$ and g_i by $g_i h_i$ we may assume that f_i and g_i vanish outside of U_i. There are two cases; $U_i \cap U_j$ is non-empty or empty.

Suppose that $U_i \cap U_j$ is non-empty. As U_i is irreducible it follows that $U_i \cap U_j$ is a dense subset of U_i. Now $f_i/g_i = f_j/g_j$ as functions on $U_i \cap U_j$ and so $f_i g_j = f_j g_i$ as functions on $U_i \cap U_j$. As these functions are continuous, $f_i g_j = f_j g_i$ on U_i. Suppose that $U_i \cap U_j$ is empty. Then the identity $f_i g_j = f_j g_i$ holds on U_i as both sides are zero.

By assumption, the common zero locus of $\{g_i\}$ is empty. Thus, by the Nullstellensatz, there are polynomials $h_1, h_2, \dots, h_n$ such that

$$1 = \sum_i g_i h_i.$$

Set $f = \sum_i f_i h_i$. I claim that the function

$$x \longrightarrow f(x),$$

is the regular function ϕ. It is enough to check this on U_j, for every j. We have

$$\begin{aligned} f g_j &= \left(\sum_i f_i g_j \right) h_i \\ &= \sum_i (f_i g_j) h_i \\ &= \sum_i (f_j g_i) h_i \\ &= f_j \sum_i g_i h_i = f_j. \end{aligned}$$

□

Note that this result implies that the working definition of a morphism between affine varieties is correct. Indeed, simply projecting onto the jth factor, it is clear that if the map is given as

$$(x_1, x_2, \dots, x_m) \longrightarrow (f_1(x), f_2(x), \dots, f_n(x)),$$

then each $f_j(x)$ is a regular function. By (5.4), it follows that $f_j(x)$ is given by a polynomial.

Lemma 5.5. *There is a contravariant functor A from the category of affine varieties over K to the category of commutative rings. Given an affine variety X we associate the ring $\mathcal{O}_X(X)$. Given a morphism*

$f\colon X \longrightarrow Y$ of affine varieties, $A(f)\colon \mathcal{O}_Y(Y) \longrightarrow \mathcal{O}_X(X)$, which sends a regular function ϕ to the regular function $A(f)(\phi) = \phi \circ f$.

It is interesting to describe the image of this functor. Clearly the ring $A(X)$ is an algebra over K (which is to say that it contains K, so that we can multiply by elements of K). Further the ring $A(X)$ is a quotient of the polynomial ring, so that it is a finitely generated algebra over K. Also since the ideal $I(X)$ is radical, the ring $A(X)$ does not have any nilpotents.

Definition 5.6. *Let R be a ring. A non-zero element r of R is said to be **nilpotent** if there is a positive integer n such that $r^n = 0$.*

Clearly if a ring has a nilpotent element, then it is not an integral domain.

Theorem 5.7. *The functor A is an equivalence of categories between the category of affine varieties over K and the category of finitely generated algebras over K, without nilpotents.*

Proof. First we show that A is essentially surjective. Suppose we are given a finitely generated algebra A over K. Pick generators $\xi_1, \xi_2, \dots, \xi_n$ of A. Define a ring homomorphism

$$\pi\colon K[x_1, x_2, \dots, x_n] \longrightarrow A,$$

simply by sending x_i to ξ_i. It is easy to check that π is an algebra homomorphism. Let I be the kernel of π. Then I is radical, as A has no nilpotents. Let $X = V(I)$. Then the coordinate ring of X is isomorphic to A, by construction. Thus A is essentially surjective.

To prove the rest, it suffices to prove that if X and Y are two affine varieties then A defines a bijection between

$$\mathrm{Hom}(X, Y) \qquad \text{and} \qquad \mathrm{Hom}(\mathcal{O}_Y(Y), \mathcal{O}_X(X)).$$

To prove this, we may as well fix embeddings $X \subset \mathbb{A}^m$ and $Y \subset \mathbb{A}^n$. In this case A naturally defines a map between

$$\mathrm{Hom}(X, Y) \qquad \text{and} \qquad \mathrm{Hom}(A(Y), A(X)),$$

which we continue to refer to as A. It suffices to prove that there is a map

$$B\colon \mathrm{Hom}(A(Y), A(X)) \longrightarrow \mathrm{Hom}(X, Y),$$

which is inverse to the map

$$A\colon \mathrm{Hom}(X, Y) \longrightarrow \mathrm{Hom}(A(Y), A(X)).$$

Suppose we are given a ring homomorphism $\alpha\colon A(Y) \longrightarrow A(X)$. Define a map

$$B(\alpha)\colon X \longrightarrow Y,$$

as follows. Let $y_1, y_2, \dots, y_n$ be coordinates on $Y \subset \mathbb{A}^n$. Let $f_1, f_2, \dots, f_n$ be the polynomials on $\mathbb{A}^n$, defined by $\alpha(y_i) = f_i$. Then define $B(\alpha)$ by the rule

$$(x_1, x_2, \dots, x_m) \longrightarrow (f_1, f_2, \dots, f_n).$$

Clearly this is a morphism. We check that the image lies in Y. Suppose that $p \in X$. We check that $q = (f_1(p), f_2(p), \dots, f_n(p)) \in Y$. Pick $g \in I(Y)$. Then

$$\begin{aligned} g(q) &= g(f_1(p), f_2(p), \dots f_n(p)) \\ &= g(\alpha(y_1)(p), \alpha(y_2)(p), \dots, \alpha(y_n)(p)) \\ &= \alpha(g)(p) \\ &= 0. \end{aligned}$$

Thus $q \in Y$ and we have defined the map B.

We now check that B is the inverse of A. Suppose that we are given a morphism $f\colon X \longrightarrow Y$. Let $\alpha = A(f)$. Suppose that f is given by $(f_1, f_2, \dots, f_n)$. Then $\alpha(y_i) = y_i \circ f = f_i$. It follows easily that $B(\alpha) = f$. Now suppose that $\alpha\colon A(Y) \longrightarrow A(X)$ is an algebra homomorphism. Then $B(\alpha)$ is given by $(f_1, f_2, \dots, f_n)$ where $f_i = \alpha(y_i)$. In this case $A(f)(y_i) = f_i$. As $y_1, y_2, \dots, y_n$ are generators of $A(Y)$, we have $\alpha = A(B(\alpha))$. $\square$

(5.7) raises an interesting question. Can we enlarge the category of affine varieties so that we get every finitely generated algebra over K and not just those without nilpotents. In fact, why stop there? Can we find a class of geometric objects, such that the space of functions on these objects, gives us any ring whatsoever (not nec. finitely generated, not nec. over K). Amazingly the answer is yes, but to do this we need the theory of schemes.