5. Coordinate rings

Recall the following version of the Nullstellensatz:

**Theorem 5.1** (Weak Nullstellensatz). Let $K$ be an algebraically closed field.

Then an ideal $m \triangleleft R = K[x_1, x_2, \ldots, x_n]$ is maximal if and only if it has the form

$$m_p = \langle x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n \rangle,$$

for some point $p = (a_1, a_2, \ldots, a_n) \in K^n$.

Note that with this formulation it is clear why we need $K$ to be algebraically closed. Indeed $I = \langle x^2 + 1 \rangle$ over $\mathbb{R}$ is in fact maximal and the vanishing locus is empty.

Another way to restate the Nullstellensatz is to observe that it establishes an inclusion reversing correspondence between ideals and closed subsets of $\mathbb{A}^n$. However this is just the tip of the iceberg.

**Definition 5.2.** Let $X \subset \mathbb{A}^n$ be a closed subset.

The **coordinate ring of** $X$, denoted $A(X)$, is the quotient

$$K[X]/I(X).$$

**Corollary 5.3.** Let $X \subset \mathbb{A}^n$ be an affine subvariety.

There is a correspondence between the points of $X$ and the maximal ideals of the coordinate ring $A(X)$.

**Proof.** Recall that there is a correspondence between ideals in $R = K[x_1, x_2, \ldots, x_n]$ containing $I$ and ideals in the quotient $R/I$. So there is a correspondence between maximal ideals of $R/I$ and maximal ideals of $R$ containing $I$.

But an ideal

$$m_p = \langle x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n \rangle,$$

contains $I$ if and only if $p \in X$ and so we are done by [5.1].

In fact this correspondence is natural. To prove this, we have to reinterpret the coordinate ring.

**Proposition 5.4.** If $X \subset \mathbb{A}^n$ is an affine variety then the ring of regular functions $\mathcal{O}_X(X)$ is isomorphic to the coordinate ring.

**Proof.** Let $\pi: K[X] \longrightarrow \mathcal{O}_X(X)$ be the map which sends a polynomial $f$ to the obvious regular function $\phi$, $\phi(x) = f(x)$. It is clear that $\pi$ is a ring homomorphism, with kernel $I(X)$. It suffices, then, to prove that $\pi$ is surjective.
Let $\phi$ be a regular function on $X$. By definition there is an open cover $U_i$ of $X$ and rational functions $f_i/g_i$ such that $\phi$ is locally given by $f_i/g_i$. As $X$ is Noetherian, we may assume that each $U_i$ is irreducible. We may assume that $U_i = U_{h_i}$ for some regular function $h_i$, as such subsets form a base for the topology. Replacing $f_i$ by $f_i h_i$ and $g_i$ by $g_i h_i$ we may assume that $f_i$ and $g_i$ vanish outside of $U_i$. There are two cases; $U_i \cap U_j$ is non-empty or empty.

Suppose that $U_i \cap U_j$ is non-empty. As $U_i$ is irreducible it follows that $U_i \cap U_j$ is a dense subset of $U_i$. Now $f_i/g_i = f_j/g_j$ as functions on $U_i \cap U_j$ and so $f_i g_j = f_j g_i$ as functions on $U_i \cap U_j$. As these functions are continuous, $f_i g_j = f_j g_i$ on $U_i$. Suppose that $U_i \cap U_j$ is empty. Then the identity $f_i g_j = f_j g_i$ holds on $U_i$ as both sides are zero.

By assumption, the common zero locus of $\{g_i\}$ is empty. Thus, by the Nullstellensatz, there are polynomials $h_1, h_2, \ldots, h_n$ such that

$$1 = \sum_i g_i h_i.$$

Set $f = \sum_i f_i h_i$. I claim that the function

$$x \mapsto f(x),$$

is the regular function $\phi$. It is enough to check this on $U_j$, for every $j$. We have

$$fg_j = \left( \sum_i f_i g_j \right) h_i$$

$$= \sum_i (f_i g_j) h_i$$

$$= \sum_i (f_j g_i) h_i$$

$$= f_j \sum_i g_i h_i = f_j. \quad \square$$

Note that this result implies that the working definition of a morphism between affine varieties is correct. Indeed, simply projecting onto the $j$th factor, it is clear that if the map is given as

$$(x_1, x_2, \ldots, x_m) \mapsto (f_1(x), f_2(x), \ldots, f_n(x)),$$

then each $f_j(x)$ is a regular function. By (5.4), it follows that $f_j(x)$ is given by a polynomial.

**Lemma 5.5.** There is a contravariant functor $A$ from the category of affine varieties over $K$ to the category of commutative rings. Given an affine variety $X$ we associate the ring $\mathcal{O}_X(X)$. Given a morphism
f : X \to Y of affine varieties, A(f): \mathcal{O}_Y(Y) \to \mathcal{O}_X(X), which sends a regular function \phi to the regular function A(f)(\phi) = \phi \circ f.

It is interesting to describe the image of this functor. Clearly the ring A(X) is an algebra over \emph{K} (which is to say that it contains \emph{K}, so that we can multiply by elements of \emph{K}). Further the ring A(X) is a quotient of the polynomial ring, so that it is a finitely generated algebra over \emph{K}. Also since the ideal I(X) is radical, the ring A(X) does not have any nilpotents.

**Definition 5.6.** Let \emph{R} be a ring. A non-zero element \emph{r} of \emph{R} is said to be \emph{nilpotent} if there is a positive integer \emph{n} such that \emph{r}^\emph{n} = 0.

Clearly if a ring has a nilpotent element, then it is not an integral domain.

**Theorem 5.7.** The functor A is an equivalence of categories between the category of affine varieties over \emph{K} and the category of finitely generated algebras over \emph{K}, without nilpotents.

**Proof.** First we show that A is essentially surjective. Suppose we are given a finitely generated algebra \emph{A} over \emph{K}. Pick generators \xi_1, \xi_2, \ldots, \xi_n of \emph{A}. Define a ring homomorphism

\[
\pi: K[x_1, x_2, \ldots, x_n] \to \emph{A},
\]

simply by sending \(x_i\) to \(\xi_i\). It is easy to check that \(\pi\) is an algebra homomorphism. Let \(I\) be the kernel of \(\pi\). Then \(I\) is radical, as \(\emph{A}\) has no nilpotents. Let \(X = V(I)\). Then the coordinate ring of \(X\) is isomorphic to \(\emph{A}\), by construction. Thus \(A\) is essentially surjective.

To prove the rest, it suffices to prove that if \(X\) and \(Y\) are two affine varieties then \(A\) defines a bijection between

\[
\text{Hom}(X, Y) \quad \text{and} \quad \text{Hom}(\mathcal{O}_Y(Y), \mathcal{O}_X(X)).
\]

To prove this, we may as well fix embeddings \(X \subset \mathbb{A}^m\) and \(Y \subset \mathbb{A}^n\). In this case \(A\) naturally defines a map between

\[
\text{Hom}(X, Y) \quad \text{and} \quad \text{Hom}(\emph{A}(Y), \emph{A}(X)),
\]

which we continue to refer to as \(A\). It suffices to prove that there is a map

\[
B: \text{Hom}(\emph{A}(Y), \emph{A}(X)) \to \text{Hom}(X, Y),
\]

which is inverse to the map

\[
A: \text{Hom}(X, Y) \to \text{Hom}(\emph{A}(Y), \emph{A}(X)).
\]

Suppose we are given a ring homomorphism \(\alpha: \emph{A}(Y) \to \emph{A}(X)\). Define a map

\[
B(\alpha): X \to Y,
\]
as follows. Let $y_1, y_2, \ldots, y_n$ be coordinates on $Y \subset \mathbb{A}^n$. Let $f_1, f_2, \ldots, f_n$ be the polynomials on $\mathbb{A}^n$, defined by $\alpha(y_i) = f_i$. Then define $B(\alpha)$ by the rule

$$(x_1, x_2, \ldots, x_m) \mapsto (f_1, f_2, \ldots, f_n).$$

Clearly this is a morphism. We check that the image lies in $Y$. Suppose that $p \in X$. We check that $q = (f_1(p), f_2(p), \ldots, f_n(p)) \in Y$. Pick $g \in I(Y)$. Then

$$g(q) = g(f_1(p), f_2(p), \ldots, f_n(p))$$

$$= g(\alpha(y_1)(p), \alpha(y_2)(p), \ldots, \alpha(y_n)(p))$$

$$= \alpha(g)(p)$$

$$= 0.$$

Thus $q \in Y$ and we have defined the map $B$.

We now check that $B$ is the inverse of $A$. Suppose that we are given a morphism $f: X \to Y$. Let $\alpha = A(f)$. Suppose that $f$ is given by $(f_1, f_2, \ldots, f_n)$. Then $\alpha(y_i) = y_i \circ f = f_i$. It follows easily that $B(\alpha) = f$. Now suppose that $\alpha: A(Y) \to A(X)$ is an algebra homomorphism. Then $B(\alpha)$ is given by $(f_1, f_2, \ldots, f_n)$ where $f_i = \alpha(y_i)$. In this case $A(f)(y_i) = f_i$. As $y_1, y_2, \ldots, y_n$ are generators of $A(Y)$, we have $\alpha = A(B(\alpha))$. \hfill \Box

(5.7) raises an interesting question. Can we enlarge the category of affine varieties so that we get every finitely generated algebra over $K$ and not just those without nilpotents. In fact, why stop there? Can we find a class of geometric objects, such that the space of functions on these objects, gives us any ring whatsoever (not nec. finitely generated, not nec. over $K$). Amazingly the answer is yes, but to do this we need the theory of schemes.