HWK #1, DUE WEDNESDAY 04/09

1. Let $\pi\colon X \longrightarrow B$ be a projective and surjective morphism with connected fibres of dimension n, where X and B are quasi-projective and X is irreducible. Let $f\colon X \longrightarrow Y$ be a morphism of quasi-projective varieties.
If there is a point $b_0 \in B$ such that $f(\pi^{-1}(b_0))$ is a point, then $f(\pi^{-1}(b))$ is a point for every $b \in B$. This result is known as *the rigidity lemma*. (*Hint: consider the morphism* $f \times \pi\colon X \longrightarrow Y \times B$).
2. Recall that an abelian variety A is a connected and projective algebraic group (you may assume that a connected algebraic group is irreducible). Show that every abelian variety is a commutative group. (*Hint: consider the morphism* $A \times A \longrightarrow A$ *given by conjugation*).
If A is a commutative algebraic group and $a \in A$ then the action of A on itself by left (or right) translation defines a morphism $\tau_a\colon A \longrightarrow A$, $\tau_a(x) = x + a$. We will refer to any such morphism as a *translation*.
3. Show that if $\pi\colon A \longrightarrow B$ is a morphism of abelian varieties then π is the composition of a translation and a group homomorphism.
4. Show that if $\pi\colon G \longrightarrow H$ is a morphism of algebraic tori then π is the composition of a translation and a group homomorphism. In particular, if $G = \mathbb{G}_m$ and $H = \mathbb{G}_m^n$ and π sends the identity to the identity then there are integers $a_1, a_2, \dots, a_n$ such that $\pi(t) = (t^{a_1}, t^{a_2}, \dots, t^{a_n})$. (*Hint: consider the map of group algebras (aka coordinate rings)*).
5. Let A be an abelian variety. Show that every rational map $f\colon \mathbb{P}^1 \dashrightarrow A$ is constant. You may use the fact that every morphism $\pi\colon G \longrightarrow A$ is a composition of a translation and a group homomorphism, where G is a group isomorphic to either $\mathbb{G}_a$ or $\mathbb{G}_m$.
(*Just for fun:* For those who know some of the theory of complex manifolds, note that if the underlying field is $\mathbb{C}$, then every abelian variety is a complex torus. Give another proof that f is constant in this case).
6. Let X and Y be two projective varieties in $\mathbb{P}^n$ of dimensions d and e. The **join** $J(X, Y)$ is the union of all lines which intersect both X and Y.
(i) Show that if X and Y belong to linear spaces which don't intersect then the dimension of the join of X and Y is equal to $d + e + 1$.
(ii) Show that if X and Y don't intersect then the dimension of the join of X and Y is equal to $d + e + 1$ (*Hint: reduce to the case above, by realising* X *and* Y *as the projection of* $\tilde{X}$ *and* $\tilde{Y}$ *in* $\mathbb{P}^{2n+1}$).

(iii) Show that if $d + e \geq n$ then X and Y must intersect.