1. Weil Divisors

Definition 1.1. *We say that a scheme X is* ***regular in codimension one*** *if every local ring of dimension one is regular, that is, the quotient $\mathfrak{m}/\mathfrak{m}^2$ is one dimensional, where $\mathfrak{m}$ is the unique maximal ideal of the corresponding local ring.*

Regular in codimension one often translates to smooth in codimension one.

When talking about Weil divisors, we will only consider schemes which are

($*$) noetherian, integral, separated, and regular in codimension one.

Definition 1.2. *Let X be a scheme satisfying ($*$). A* ***prime divisor*** *Y on X is a closed integral subscheme of codimension one.*

A ***Weil divisor*** *D on X is an element of the free abelian group $\operatorname{Div} X$ generated by the prime divisors.*

Thus a Weil divisor is a formal linear combination $D = \sum_Y n_Y Y$ of prime divisors, where all but finitely many $n_Y = 0$. We say that D is **effective** if $n_Y \geq 0$.

Definition 1.3. *Let X be a scheme satisfying ($*$), and let Y be a prime divisor, with generic point η. Then $\mathcal{O}_{X,\eta}$ is a discrete valuation ring with quotient field K.*

The ***valuation*** *ν_Y* ***associated to*** *Y is the corresponding valuation.*

Note that as X is separated, Y is determined by its valuation. If $f \in K = K(X)$ and $\nu_Y(f) > 0$ then we say that f has a **zero of order** $\nu_Y(f)$; if $\nu_Y(f) < 0$ then we say that f has a **pole of order** $-\nu_Y(f)$.

Definition-Lemma 1.4. *Let X be a scheme satisfying ($*$), and let $f \in K^*$.*

$$(f) = \sum_Y \nu_Y(f) Y \in \operatorname{Div} X.$$

Proof. We have to show that $\nu_Y(f) = 0$ for all but finitely many Y. Let U be the open subset where f is regular. Then the only poles of f are along $Z = X - U$. As Z is a proper closed subset and X is noetherian, Z contains only finitely many prime divisors.

Similarly the zeroes of f only occur outside the open subset V where $g = f^{-1}$ is regular. $\square$

Any divisor D of the form (f) will be called **principal**.

Lemma 1.5. *Let X be a scheme satisfying $(*)$.*
The principal divisors are a subgroup of $\operatorname{Div} X$*.*

Proof. The map

$$K^* \longrightarrow \operatorname{Div} X,$$

is easily seen to be a group homomorphism. □

Definition 1.6. *Two Weil divisors D and D' are called **linearly equivalent**, denoted $D \sim D'$, if and only if the difference is principal. The group of Weil divisors modulo linear equivalence is called the **divisor Class group**, denoted* $\operatorname{Cl} X$*.*

We will also denote the group of Weil divisors modulo linear equivalence as $A_{n-1}(X)$.

Proposition 1.7. *If k is a field then*

$$\operatorname{Cl}(\mathbb{P}^n_k) \simeq \mathbb{Z}.$$

Proof. Note that if Y is a prime divisor in $\mathbb{P}^n_k$ then Y is a hypersurface in $\mathbb{P}^n$, so that $I = \langle G \rangle$ and Y is defined by a single homogeneous polynomial G. The degree of G is called the degree of Y.

If $D = \sum n_Y Y$ is a Weil divisor then define the degree $\deg D$ of D to be the sum

$$\sum n_Y \deg Y,$$

where $\deg Y$ is the degree of Y.

Note that the degree of any rational function is zero. Thus there is a well-defined group homomorphism

$$\deg\colon \operatorname{Cl}(\mathbb{P}^r_k) \longrightarrow \mathbb{Z},$$

and it suffices to prove that this map is an isomorphism. Let H be defined by X_0. Then H is a hyperplane and H has degree one. The divisor $D = nH$ has degree n and so the degree map is surjective. One the other hand, if $D = \sum n_i Y_i$ is effective, and Y_i is defined by G_i,

$$(\prod_i G^{n_i}/X_0^d) = D - dH,$$

where d is the degree of D, so that $D \sim dH$. □

The next case, at least over an algebraically closed field is a smooth cubic curve in $\mathbb{P}^2_k$. We will need:

Theorem 1.8. *A smooth cubic curve is always irrational.*

Example 1.9. *Let C be a smooth cubic curve in $\mathbb{P}^2_k$. Suppose that the line $Z = 0$ is a flex line to the cubic at the point $P_0 = [0 : 1 : 0]$. If the equation of the cubic is $F(X, Y, Z)$ this says that $F(X, Y, 0) = X^3$. Therefore the cubic has the form $X^3 + ZG(X, Y, Z)$. If we work on the open subset $U_3 \simeq \mathbb{A}^2_k$, then we get*

$$x^3 + g(x, y) = 0,$$

where $g(x, y)$ has degree at most two. If we expand $g(x, y)$ as a polynomial in y,

$$g_0(x)y^2 + g_1(x)y + g_2(x),$$

then $g_0(x)$ must be a non-zero scalar, since otherwise C is singular (a nodal or cuspidal cubic). We may assume that $g_0 = 1$. If we assume that the characteristic is not two then we may complete the square to get

$$y^2 = x^3 + g(x),$$

for some quadratic polynomial $g(x)$. If we assume that the characteristic is not three then we may complete the cube to get

$$y^2 = x^3 + ax + b,$$

for some a and $b \in k$.

Now any two sets of three collinear points are linearly equivalent (since the equation of one line divided by another line is a rational function on the whole $\mathbb{P}^2_k$). In fact given any three points P, Q and P' we may find Q' such that $P + Q \sim P' + Q'$; indeed the line $l = \langle P, Q\rangle$ meets the cubic in one more point R. The line $l' = \langle R, P'\rangle$ then meets the cubic in yet another point Q'. We have

$$P + Q + R \sim P' + Q' + R'.$$

Cancelling we get

$$P + Q \sim P' + Q'.$$

It follows that if there are further linear equivalences then there are two points P and P' such that $P \sim P'$. This gives us a rational function f with a single zero P and a single pole P'; in turn this gives rise to a morphism $C \longrightarrow \mathbb{P}^1$ which is an isomorphism. It turns out that a smooth cubic is not isomorphic to $\mathbb{P}^1$, so that in fact the only relations are those generated by setting two sets of three collinear points to be linearly equivalent.

Put differently, the rational points of C form an abelian group, where three points sum to zero if and only if they are collinear, and P_0 is declared to be the identity. The divisors of degree zero modulo linear equivalence are equal to this group.

In particular, an elliptic curve is very far from being isomorphic to $\mathbb{P}^1_k$.

It is interesting to calculate the Class group of a toric variety X, which always satisfies $(*)$. By assumption there is a dense open subset $U \simeq \mathbb{G}^n_m$. The complement Z is a union of the invariant divisors.

Lemma 1.10. *Suppose that X satisfies $(*)$, let Z be a closed subset and let $U = X \setminus Z$.*

Then there is an exact sequence

$$\mathbb{Z}^k \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U) \longrightarrow 0,$$

where k is the number of components of Z which are prime divisors.

Proof. If Y is a prime divisor on X then $Y' = Y \cap U$ is either a prime divisor on U or empty. This defines a group homomorphism

$$\rho\colon \mathrm{Div}(X) \longrightarrow \mathrm{Div}(U).$$

If $Y' \subset U$ is a prime divisor then let Y be the closure of Y' in X. Then Y is a prime divisor and $Y' = Y \cap U$. Thus ρ is surjective. If f is a rational function on X and $Y = (f)$ then the image of Y in $\mathrm{Div}(U)$ is equal to $(f|_U)$. If $Z = Z' \cup \bigcup_{i=1}^k Z_i$ where Z' has codimension at least two then the map which sends $(m_1, m_2, \dots, m_k)$ to $\sum m_i Z_i$ generates the kernel. □

Example 1.11. *Let $X = \mathbb{P}^2_k$ and C be an irreducible curve of degree d. Then $\mathrm{Cl}(\mathbb{P}^2 - C)$ is equal to $\mathbb{Z}_d$. Similarly $\mathrm{Cl}(\mathbb{A}^n_k) = 0$.*

It follows by (1.10) that there is an exact sequence

$$\mathbb{Z}^k \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U) \longrightarrow 0.$$

Applying this to $X = \mathbb{A}^n_k$ it follows that $\mathrm{Cl}(U) = 0$. So we get an exact sequence

$$0 \longrightarrow K \longrightarrow \mathbb{Z}^s \longrightarrow \mathrm{Cl}(X) \longrightarrow 0.$$

We want to identify the kernel. This is equal to the set of principal divisors which are supported on the invariant divisors. If f is a rational function such that (f) is supported on the invariant divisors then f has no zeroes or poles on the torus; it follows that $f = \lambda\chi^u$, where $\lambda \in k^*$ and $u \in M$.

It follows that there is an exact sequence

$$M \longrightarrow \mathbb{Z}^s \longrightarrow \mathrm{Cl}(X) \longrightarrow 0.$$

Lemma 1.12. *Let $u \in M$. Suppose that X is the affine toric variety associated to a cone σ, where σ spans $N_{\mathbb{R}}$. Let v be a primitive generator of a one dimensional ray τ of σ and let D be the corresponding invariant divisor.*

Then $\operatorname{ord}_D(\chi^u) = \langle u, v\rangle$. *In particular*

$$(\chi^u) = \sum_i \langle u, v_i\rangle D_i,$$

where the sum ranges over the invariant divisors.

Proof. We can calculate the order on the open set $U_\tau = \mathbb{A}^1_k \times \mathbb{G}_m^{n-1}$, where D corresponds to $\{0\} \times \mathbb{G}_m^{n-1}$. Using this, we are reduced to the one dimensional case. So $N = \mathbb{Z}$, $v = 1$ and $u \in M = \mathbb{Z}$. In this case χ^u is the monomial x^u and the order of vanishing at the origin is exactly u. $\square$

It follows that if $X = X(F)$ is the toric variety associated to a fan F which spans $N_{\mathbb{R}}$ then we have a short exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^s \longrightarrow \operatorname{Cl}(X) \longrightarrow 0.$$

Example 1.13. *Let σ be the cone spanned by $2e_1 - e_2$ and e_2 inside $N_{\mathbb{R}} = \mathbb{R}^2$. There are two invariant divisors D_1 and D_2. The principal divisor associated to $u = f_1 = (1,0)$ is $2D_1$ and the principal divisor associated to $u = f_2 = (0,1)$ is $D_2 - D_1$. So the class group is $\mathbb{Z}_2$.*

Note that the dual $\check{\sigma}$ is the cone spanned by f_1 and $f_1 + 2f_2$. Generators for the monoid $S_\sigma = \check{\sigma} \cap M$ are f_1, $f_1 + f_2$ and $f_1 + 2f_2$. So the group algebra

$$A_\sigma = k[x, xy, xy^2] = \frac{k[u,v,w]}{\langle v^2 - uw\rangle},$$

and $X = U_\sigma$ is the quadric cone.

Now suppose we take the standard fan associated to $\mathbb{P}^2$. The invariant divisors are the three coordinate lines, D_1, D_2 and D_3. If $f_1 = (1,0)$ and $f_2 = (0,1)$ then

$$(\chi^{f_1}) = D_1 - D_3 \qquad \text{and} \qquad (\chi^{f_2}) = D_2 - D_3.$$

So the class group is $\mathbb{Z}$.