12. Flips and flops

It is interesting to figure out the geometry behind the example of a toric variety which is not projective. To warm up, suppose that we start with $\mathbb{A}^3_k$. This is the toric variety associated to the fan spanned by e_1, e_2, e_3. Imagine blowing up two of the axes. This corresponds to inserting two vectors, $e_1 + e_2$ and $e_1 + e_3$. However the order in which we blow up is significant. Let's introduce some notation. If we blow up the x-axis $\pi\colon Y \longrightarrow X$ and then the y-axis, $\psi\colon Z \longrightarrow Y$, let's call the exceptional divisors E_1 and E_2, and let E_1' denote the strict transform of E_1 on Z. E_1 is a $\mathbb{P}^1$-bundle over the x-axis. The strict transform of the y-axis in Y intersects E_1 in a point p. When we blow up this curve, $E_1' \longrightarrow E_1$ blows up the point p. The fibre of E_1' over the origin therefore consists of two copies Σ_1 and Σ_2 of $\mathbb{P}^1$. Σ_1 is the strict transform of the fibre of E_1 over the origin and Σ_2 is the exceptional divisor. The fibre Σ of E_2 over the origin is a copy of $\mathbb{P}^1$. Σ and Σ_2 are the same curve in Z.

The example of a toric variety which is not projective is obtained from $\mathbb{P}^3$ by blowing up three coordinate axes, which form a triangle. The twist is that we do something different at each of the three coordinate points. Suppose that $\pi\colon X \longrightarrow \mathbb{P}^3$ is the birational morphism down to $\mathbb{P}^3$, and let E_1, E_2 and E_3 be the three exceptional divisors. Over one point we extract E_1 first then E_2, over the second point we extract first E_2 then E_3 and over the last point we extract first E_3 then E_1.

To see what has gone wrong, we need to work in the homology and cohomology groups of X. Any curve C in X determines an element of $[C] \in H_2(X, \mathbb{Z})$. Any Cartier divisor D in X determines a class $[D] \in H^2(X, \mathbb{Z})$. We can pair these two classes to get an intersection number $D \cdot C \in \mathbb{Z}$. One way to compute this number is to consider the line bundle $\mathcal{L} = \mathcal{O}_X(D)$ associated to D. Then

$$D \cdot C = \deg \mathcal{L}|_C.$$

If D is ample then this intersection number is always positive. This implies that the class of every curve is non-trivial in homology.

Suppose the reducible fibres of E_1, E_2 and E_3 over their images are $A_1 + A_2$, $B_1 + B_2$ and $C_1 + C_3$. Suppose that the general fibres are A, B and C. We suppose that A_1 is attached to B, B_1 is attached to C

and C_1 is attached to A. We have

$$\begin{aligned}
[A] &= [A_1] + [A_2] \\
&= [B] + [A_2] \\
&= [B_1] + [B_2] + [A_2] \\
&= [C] + [B_2] + [A_2] \\
&= [C_1] + [C_2] + [B_2] + [A_2] \\
&= [A] + [C_2] + [B_2] + [A_2],
\end{aligned}$$

in $H_2(X, \mathbb{Z})$, so that

$$[A_2] + [B_2] + [C_2] = 0 \in H_2(X, \mathbb{Z}).$$

Suppose that D were an ample divisor on X. Then

$$0 = D \cdot ([A_2] + [B_2] + [C_2]) = D \cdot [A_2] + D \cdot [B_2] + D \cdot [C_2] > 0,$$

a contradiction.

There are a number of things to say about this way of looking at things, which lead in different directions. The first is that there is no particular reason to start with a triangle of curves. We could start with two conics intersecting transversally (so that they lie in different planes). We could even start with a nodal cubic, and just do something different over the two branches of the curve passing through the node. Neither of these examples are toric, of course. It is clear that in the first two examples, the morphism

$$\pi \colon X \longrightarrow \mathbb{P}^3,$$

is locally projective. It cannot be a projective morphism, since $\mathbb{P}^3$ is projective and the composition of projective is projective. It also follows that π is not the blow up of a coherent sheaf of ideals on $\mathbb{P}^3$. The third example is not even a variety. It is a complex manifold (and in fact it is something called an algebraic space). In particular the notion of the blow up in algebraic geometry is more delicate than it might first appear.

The second thing is to consider the difference between the order of blow ups of the two axes. Suppose we denote the composition of blowing up the x-axis and then the y-axis by $\pi_1 \colon X_1 \longrightarrow \mathbb{A}^3$ and the composition the other way by $\pi_2 \colon X_2 \longrightarrow \mathbb{A}^3$. Now X_1 and X_2 agree outside the origin. Let $\phi \colon X_1 \dashrightarrow X_2$ be the resulting birational map. If $A_1 + A_2$ is the fibre of π_1 over the origin and $B_1 + B_2$ is the fibre of π_2 over the origin, then ϕ is in fact an isomorphism outside A_2 and B_2. So ϕ is a birational map which is an isomorphism in codimension one, in fact an isomorphism outside a curve, isomorphic to $\mathbb{P}^1$. ϕ is an example of a *flop*. In terms of fans, we have four vectors v_1, v_2, v_3 and v_4, such that

$$v_1 + v_3 = v_2 + v_4,$$

and any three vectors span the lattice. If σ is the cone spanned by these four vectors, then $Q = U_\sigma$ is the cone over a quadric. There are two ways to subdivide σ into two cones. Insert the edge connecting v_1 to v_3 or the edge corresponding to $v_2 + v_4$. The corresponding morphisms extract a copy of $\mathbb{P}^1$ and the resulting birational map between the two toric varieties is a (simple) flop. One can also insert the vector $w = v_1 + v_3$, to get a toric variety Y. The corresponding exceptional divisor is $\mathbb{P}^1 \times \mathbb{P}^1$. These toric varieties fit into a picture

$$\begin{tikzcd}
 & Y \arrow[dl] \arrow[dr] & \\
X_1 \arrow[rr, dashed] \arrow[dr] & & X_2 \arrow[dl] \\
 & Q. &
\end{tikzcd}$$

The two maps $Y \longrightarrow X_i$ correspond to the two projections of $\mathbb{P}^1 \times \mathbb{P}^1$ down to $\mathbb{P}^1$. Now we know that $\pi_i \colon X_i \longrightarrow Q$ corresponds to blowing up a coherent ideal sheaf. In fact it corresponds to blowing up a Weil divisor (in fact this is a given, as π_i does not extract any divisors), the plane determined by either ruling.

Finally, it is interesting to wonder more about the original examples of varieties which are not projective. Note that in the case when we blow up either a triangle or a conic if we make one flop then we get a projective variety. Put differently, if we start with a projective variety then it is possible to get a non-projective variety by flopping a curve. When does flopping a curve mean that the variety is no longer projective? A variety is projective if it contains an ample divisor. Ample divisors intersect all curve positively. Note that any sum of ample divisors is ample.

Definition 12.1. *Let X be a proper variety. The **ample cone** is the cone in $H^2(X, \mathbb{R})$ spanned by the classes of the ample divisors.*

*The **Kleiman-Mori cone** $\overline{\mathrm{NE}}(X)$ in $H_2(X, \mathbb{R})$ is the closure of the cone spanned by the classes of curves.*

The significance of all of this is the following:

Theorem 12.2 (Kleiman's Criteria)**.** *Let X be a proper variety (or even algebraic space).*

A divisor D is ample if and only if the linear functional

$$\psi\colon H_2(X,\mathbb{R}) \longrightarrow \mathbb{R},$$

given by $\phi(\alpha) = [D]\cdot\alpha$ is strictly positive on $\overline{\mathrm{NE}}(X) - \{0\}$.

Using Kleiman's criteria, it is not hard to show that if $\phi\colon X \dashrightarrow Y$ is a flop of the curve C and X is projective then Y is projective if and only if the class of $[C]$ generates a one dimensional face of $\overline{\mathrm{NE}}(X)$.