13. Kähler differentials

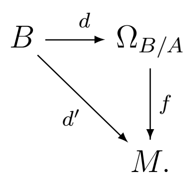
Let A be a ring, B an A-algebra and M a B-module.

Definition 13.1. *An A-**derivation** of B into M is a map* $\mathrm{d}\colon B \longrightarrow M$ *such that*

(1) $\mathrm{d}(b_1 + b_2) = \mathrm{d}b_1 + \mathrm{d}b_2$.
(2) $\mathrm{d}(bb') = b'\mathrm{d}b + b\mathrm{d}b'$.
(3) $\mathrm{d}a = 0$.

Definition 13.2. *The module of **relative differentials**, denoted $\Omega_{B/A}$, is a B-module together with an A-derivation,* $\mathrm{d}\colon B \longrightarrow \Omega_{B/A}$, *which is universal with this property:*

If M is a B-module and $\mathrm{d}'\colon B \longrightarrow M$ is an A-derivation then there exists a unique B-module homomorphism $f\colon \Omega_{B/A} \longrightarrow M$ which makes the following diagram commute:



One can construct the module of relative differentials in the usual way; take the free B-module, with generators

$$\{\, \mathrm{d}b \mid b \in B \,\},$$

and quotient out by the three obvious sets of relations

(1) $\mathrm{d}(b_1 + b_2) - \mathrm{d}b_1 - \mathrm{d}b_2$,
(2) $\mathrm{d}(bb') - b'\mathrm{d}b - b\mathrm{d}b'$, and
(3) $\mathrm{d}a$.

The map $\mathrm{d}\colon B \longrightarrow M$ is the obvious one.

Example 13.3. *Let $B = A[x_1, x_2, \dots, x_n]$. Then $\Omega_{B/A}$ is the free B-module generated by $\mathrm{d}x_1, \mathrm{d}x_2, \dots, \mathrm{d}x_n$.*

Proposition 13.4. *Let A' and B be A-algebras and $B' = B \underset{A}{\otimes} A'$. Then*

$$\Omega_{B'/A'} = \Omega_{B'/A} \underset{B}{\otimes} B'$$

Furthermore, if S is a multiplicative system in B, then

$$\Omega_{S^{-1}B/A} = S^{-1}\Omega_{B/A}.$$

Suppose that $X = \operatorname{Spec} B \longrightarrow Y = \operatorname{Spec} A$ is a morphism of affine schemes. The **sheaf of relative differentials** $\Omega_{X/Y}$ is the quasi-coherent sheaf associated to the module of relative differentials $\Omega_{B/A}$.

Example 13.5. *Let* $X = \operatorname{Spec}\mathbb{R}$ *and* $Y = \operatorname{Spec}\mathbb{Q}$*. Then* $\mathrm{d}\pi \in \Omega_{X/Y} = \Omega_{\mathbb{R}/\mathbb{Q}}$ *is a non-zero differential.*

One could use the affine case to construct the sheaf of relative differentials globally. A better way to proceed is to use a little bit more algebra (and geometric intuition):

Proposition 13.6. *Let* B *be an* A*-algebra. Let*

$$B \underset{A}{\otimes} B \longrightarrow B,$$

be the diagonal morphism $b \otimes b' \longrightarrow bb'$ *and let* I *be the kernel. Consider* $B \underset{A}{\otimes} B$ *as a* B*-module by multiplication on the left. Then* I/I^2 *inherits the structure of a* B*-module. Define a map*

$$\mathrm{d}\colon B \longrightarrow \frac{I}{I^2},$$

by the rule

$$\mathrm{d}b = 1 \otimes b - b \otimes 1.$$

Then I/I^2 *is the module of differentials.*

Now suppose we are given a morphism of schemes $f\colon X \longrightarrow Y$. This induces the diagonal morphism

$$\Delta\colon X \longrightarrow X \underset{Y}{\times} X.$$

Then Δ defines an isomorphism of X with its image $\Delta(X)$ and this is locally closed in $X \underset{Y}{\times} X$, that is, there is an open subset $W \subset X \underset{Y}{\times} X$ and $\Delta(X)$ is a closed subset of W (it is closed in $X \underset{Y}{\times} X$ if and only if X is separated).

Definition 13.7. *Let* $\mathcal{I}$ *be the sheaf of ideals of* $\Delta(X)$ *inside* W*. The* ***sheaf of relative differentials***

$$\Omega_{X/Y} = \Delta^*\left(\frac{\mathcal{I}}{\mathcal{I}^2}\right).$$

Theorem 13.8 (Euler sequence)**.** *Let* A *be a ring, let* $Y = \operatorname{Spec} A$ *and* $X = \mathbb{P}^n_A$.

Then there is a short exact sequence of $\mathcal{O}_X$*-modules*

$$0 \longrightarrow \Omega_{X/Y} \longrightarrow \mathcal{O}_X(-1)^{n+1} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Proof. Let $S = A[x_0, x_1, \dots, x_n]$ be the homogeneous coordinate ring of X. Let E be the graded S-module $S(-1)^{n+1}$, with basis $e_0, e_1, \dots, e_n$ in degree one. Define a (degree 0) homomorphism of graded S-modules

$E \longrightarrow S$ by sending $e_i \longrightarrow x_i$ and let M be the kernel. We have a left exact sequence

$$0 \longrightarrow M \longrightarrow E \longrightarrow S.$$

This gives rise to a short exact sequence of $\mathcal{O}_X$-modules,

$$0 \longrightarrow \tilde{M} \longrightarrow \mathcal{O}_X(-1)^{n+1} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Note that even though $E \longrightarrow S$ is not surjective, it is surjective in all non-negative degrees, so that the map of sheaves is surjective.

It remains to show that $\tilde{M} \simeq \Omega_{X/Y}$. First note that if we localise at x_i, then $E_{x_i} \longrightarrow S_{x_i}$ is a surjective homomorphism of free S_{x_i}-modules, so that M_{x_i} is a free S_{x_i}-module of rank n, generated by

$$\{ e_j - \frac{x_j}{x_i} e_i \,|\, j \neq i \}.$$

It follows that if U_i is the standard open affine subset of X defined by x_i then $\tilde{M}|_{U_i}$ is a free $\mathcal{O}_{U_i}$-module of rank n generated by the sections

$$\{ \frac{1}{x_i} e_j - \frac{x_j}{x_i^2} e_i \,|\, j \neq i \}.$$

(We need the extra factor of x_i to get elements of degree zero.)

We define a map

$$\phi_i \colon \Omega_{X/Y}|_{U_i} \longrightarrow \tilde{M}|_{U_i},$$

as follows. As $U_i = \operatorname{Spec} k[\frac{x_0}{x_i}, \frac{x_1}{x_i}, \dots, \frac{x_n}{x_i}]$, it follows that $\Omega_{X/Y}$ is the free $\mathcal{O}_{U_i}$-module generated by

$$\mathrm{d}\left(\frac{x_0}{x_i}\right), \mathrm{d}\left(\frac{x_1}{x_i}\right), \dots, \mathrm{d}\left(\frac{x_n}{x_i}\right).$$

So we define ϕ_i by the rule

$$\mathrm{d}\left(\frac{x_j}{x_i}\right) \longrightarrow \frac{1}{x_i} e_j - \frac{x_j}{x_i^2} e_i.$$

ϕ_i is clearly an isomorphism. We check that we can glue these maps to a global isomorphism,

$$\phi \colon \Omega_{X/Y} \longrightarrow \tilde{M}.$$

On $U_i \cap U_j$, we have

$$\left(\frac{x_k}{x_i}\right) = \left(\frac{x_k}{x_j}\right)\left(\frac{x_j}{x_i}\right).$$

Hence in $(\Omega_{X/Y})|_{U_i \cap U_j}$ we have

$$\mathrm{d}\left(\frac{x_k}{x_i}\right) - \frac{x_k}{x_j}\mathrm{d}\left(\frac{x_j}{x_i}\right) = \frac{x_j}{x_i}\mathrm{d}\left(\frac{x_k}{x_j}\right).$$

If we apply ϕ_i to the LHS and ϕ_j to the RHS, we get the same thing, namely

$$\frac{1}{x_i x_j} \left(x_j e_k - x_k e_j \right).$$

Thus the isomorphisms ϕ_i glue together. $\square$