14. Smoothness

Definition 14.1. *A variety is **smooth** (aka non-singular) if all of its local rings are regular local rings.*

Theorem 14.2. *The localisation of any regular local ring at a prime ideal is a regular local ring.*

Thus to check if a variety is smooth it is enough to consider only the closed points.

Theorem 14.3. *Let X be an irreducible separate scheme of finite type over an algebraically closed field k.*

Then $\Omega_{X/k}$ is locally free of rank $n = \dim X$ if and only if X is a smooth variety over k.

If $X \longrightarrow Z$ is a morphism of schemes and $Y \subset X$ is a closed subscheme, with ideal sheaf $\mathcal{I}$. Then there is an exact sequence of sheaves on Z,

$$\frac{\mathcal{I}}{\mathcal{I}^2} \longrightarrow \Omega_{X/Z} \otimes \mathcal{O}_Y \longrightarrow \Omega_{Y/Z} \longrightarrow 0.$$

Theorem 14.4. *Let X be a smooth variety of dimension n. Let $Y \subset X$ be an irreducible closed subscheme with sheaf of ideals $\mathcal{I}$.*

Then Y is smooth if and only if

(1) $\Omega_{Y/k}$ is locally free, and
(2) the sequence above is also left exact:

$$0 \longrightarrow \frac{\mathcal{I}}{\mathcal{I}^2} \longrightarrow \Omega_{X/k} \otimes \mathcal{O}_Y \longrightarrow \Omega_{Y/k} \longrightarrow 0.$$

Furthermore, in this case, $\mathcal{I}$ is locally generated by $r = \operatorname{codim}(Y, X)$ elements and $\frac{\mathcal{I}}{\mathcal{I}^2}$ is locally free of rank r on Y.

Proof. Suppose (1) and (2) hold. Then $\Omega_{Y/k}$ is locally free and so we only have to check that its rank q is equal to the dimension of Y. Then $\mathcal{I}/\mathcal{I}^2$ is locally free of rank $n - q$. Nakayama's lemma implies that $\mathcal{I}$ is locally generated by $n-q$ elements and so $\dim Y \geq n-(n-q) = q$. On the other hand, if $y \in Y$ is any closed point $q = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$ and so $q \geq \dim Y$. Thus $q = \dim Y$. This also establishes the last statement.

Now suppose that Y is smooth. Then $\Omega_{Y/k}$ is locally free of rank $q = \dim Y$ and so (1) is immediate. On the other hand, there is an exact sequence

$$\frac{\mathcal{I}}{\mathcal{I}^2} \longrightarrow \Omega_{X/k} \otimes \mathcal{O}_Y \longrightarrow \Omega_{Y/k} \longrightarrow 0.$$

Pick a closed point $y \in Y$. As $\mathcal{I}/\mathcal{I}^2$ is locally free of rank $r = n - q$, we may pick sections $x_1, x_2, \dots, x_r$ of $\mathcal{I}$ such that $\mathrm{d}x_1, \mathrm{d}x_2, \dots, \mathrm{d}x_r$ generate the kernel of the second map.

Let $Y' \subset X$ be the corresponding closed subscheme. Then, by construction, $\mathrm{d}x_1, \mathrm{d}x_2, \dots, \mathrm{d}x_r$ generate a free subsheaf of rank r of $\Omega_{X/k} \otimes \mathcal{O}_{Y'}$ in a neighbourhood of y. It follows that for the exact sequence for Y'

$$\frac{\mathcal{I}'}{\mathcal{I}'^2} \longrightarrow \Omega_{X/Z} \otimes \mathcal{O}_Y \longrightarrow \Omega_{Y'/Z} \longrightarrow 0,$$

the first map is injective and $\Omega_{Y'/k}$ is locally free of rank $n - r$. But then Y' is smooth and $\dim Y' = \dim Y$. As $Y \subset Y'$ and Y' is integral, we must have $Y = Y'$ and this gives (2). ☐

Theorem 14.5 (Bertini's Theorem)**.** *Let $X \subset \mathbb{P}^n_k$ be a closed smooth projective variety. Then there is a hyperplane $H \subset \mathbb{P}^n_k$, not containing X, such that $H \cap X$ is regular at every point.*

Furthermore the set of such hyperplanes forms an open dense subset of the linear system $|H| \simeq \mathbb{P}^n_k$.

Proof. Let $x \in X$ be a closed point. Call a hyperplane H **bad** if either H contains X or H does not contain X but it does contain x and $X \cap H$ is not regular at x. Let B_x be the set of all bad hyperplanes at x. Fix a hyperplane H_0 not containing x, defined by $f_0 \in V = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Define a map

$$\phi_x \colon V \longrightarrow \mathcal{O}_{X,x}/\mathfrak{m}^2,$$

as follows. Given by $f \in V$, f/f_0 is a regular function on $X - X \cap X_0$. Send f to the image of f/f_0 to its class in the quotient $\mathcal{O}_{X,x}/\mathfrak{m}^2$. Now $x \in X \cap H$ if and only if $\phi_x(f) \in \mathfrak{m}$ and $x \in X \cap H$ is a regular point if and only if $\phi_x(f) \neq 0$.

Thus B_x is precisely the kernel of ϕ_x. Now as k is algebraically closed and x is a closed point, ϕ_x is surjective. If $\dim X = r$ then $\mathcal{O}_{X,x}/\mathfrak{m}^2$ has dimension $r + 1$ and so B_x is a linear subspace of $|H|$ of dimension $n - r - 1$.

Let $B \subset X \times |H|$ be the set of pairs (x, H) where $H \in B_x$. Then B is a closed subset. Let $p \colon B \longrightarrow X$ and $q \colon B \longrightarrow |H|$ denote projection onto either factor. p is surjective, with irreducible fibres of dimension $n - r - 1$. It follows that B is irreducible of dimension $r + (n - r - 1) = n - 1$. The image of q has dimension at most $n - 1$. Hence $q(B)$ is a proper closed subset of $|H|$. ☐

Remark 14.6. *We will see later that $H \cap X$ is in fact connected, whence irreducible, so that in fact $Y = H \cap X$ is a smooth subvariety.*

Definition 14.7. *Let X be a smooth variety. The* ***tangent sheaf***

$$\mathcal{T}_X = \mathbf{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X).$$

Note that the tangent sheaf is a locally free sheaf of rank equal to the dimension of X.