16. Homological algebra and derived functors

Let X be any analytic space. Then there is an exact sequence of sheaves

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X^{\mathrm{an}} \longrightarrow \mathcal{O}_X^* \longrightarrow 0,$$

where $\mathcal{O}_X^*$ is the sheaf of nowhere zero holomorphic functions under multiplication. The map

$$\mathcal{O}_X^{\mathrm{an}} \longrightarrow \mathcal{O}_X^*,$$

sends a holomorphic function f to $e^{2\pi i f}$, so that the kernel is clearly $\mathbb{Z}$, the sheaf of locally constant, integer valued functions. Given a nowhere zero holomorphic function g, locally we can always find a function f mapping to g, since locally we can always take logs. Thus the map of sheaves is surjective as it is surjective on stalks.

Now suppose we take global sections. We get an exact sequence

$$0 \longrightarrow H^0(X, \mathbb{Z}) \longrightarrow H^0(X, \mathcal{O}_X^{\mathrm{an}}) \longrightarrow H^0(X, \mathcal{O}_X^*),$$

but in general the last map is not surjective.

For example, let $X = \mathbb{C} - \{ 0 \}$. Then z is a nowhere zero function which is not the exponential of any holomorphic function; the logarithm is not a globally well-defined function on the whole punctured plane.

Sheaf cohomology is introduced exactly to fix lack of exactness on the left.

Definition 16.1. *An* ***abelian category*** $\mathcal{U}$ *is a category such that for every pair of objects A and B,* $\mathrm{Hom}(A, B)$ *has the structure of an abelian group and the composition law is linear; finite direct sums exist; every morphism has a kernel and a cokernel; every monomorphism is the kernel of its cokernel, every epimorphism is the cokernel of its kernel; and every morphism can be factored into an epimorphism followed by a monomorphism.*

Example 16.2. *Here are some examples of abelian categories:*

1. *The category of abelian groups.*
2. *The category of modules over a ring A.*
3. *The category of sheaves of abelian groups on a topological space X.*
4. *The category of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$.*
5. *The category of quasi-coherent sheaves of $\mathcal{O}_X$-modules on a scheme X.*
6. *The category of coherent sheaves of $\mathcal{O}_X$-modules on a noetherian scheme X.*

A **complex** $A^\bullet$ of objects in an abelian category is a sequence of objects, indexed by $\mathbb{Z}$, together with coboundary maps

$$d^i \colon A^i \longrightarrow A^{i+1},$$

such that the composition of any two is zero.

The ith **cohomology** of the complex, is obtained in the usual way:

$$h^i(A^\bullet) = \operatorname{Ker} d^i / \operatorname{Im} d^{i+1}.$$

A morphism of complexes $f \colon A^\bullet \longrightarrow B^\bullet$ is simply a collection of morphisms $f^i \colon A^i \longrightarrow B^i$ compatible with the coboundary maps. They give rise to maps

$$h^i(f) \colon h^i(A^\bullet) \longrightarrow h^i(B^\bullet).$$

Two morphisms f and g are **homotopic** if there are maps $k^i \colon A^i \longrightarrow B^{i-1}$ such that

$$f - g = dk + kd.$$

If f and g are homotopic then $h^i(f) = h^i(g)$. Two complexes are **homotopic** if there are maps $f \colon A^\bullet \longrightarrow B^\bullet$ and $g \colon B^\bullet \longrightarrow A^\bullet$ whose composition either way is homotopic to the identity.

A functor F from one abelian category $\mathcal{U}$ to another $\mathcal{B}$ is called **additive** if for any two objects A and B in $\mathcal{U}$, the induced map

$$\operatorname{Hom}(A, B) \longrightarrow \operatorname{Hom}(FA, FB),$$

is a group homomorphism. F is **left exact** if in addition, given an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

we get an exact sequence,

$$0 \longrightarrow FA \longrightarrow FB \longrightarrow FC.$$

Example 16.3. *Fix an object T. The functor*

$$A \longrightarrow \operatorname{Hom}(A, T),$$

is (contravariant) left exact, so that we have an exact sequence

$$0 \longrightarrow \operatorname{Hom}(C, T) \longrightarrow \operatorname{Hom}(B, T) \longrightarrow \operatorname{Hom}(A, T).$$

An object I of $\mathcal{U}$ is **injective** if the functor

$$A \longrightarrow \operatorname{Hom}(A, I),$$

is exact. An **injective resolution** of an object A is a complex $I^\bullet$, such that I is zero in negative degrees, I^i is an injective object and there is a morphism $A \longrightarrow I^0$ such that the obvious complex is exact.

We say that $\mathcal{U}$ has **enough injectives** if every object embeds into an injective object. In this case, every object has an injective resolution and any two such are homotopic.

Given a category with enough injectives, define the **right derived functors** of a left exact functor F by fixing an injective resolution and then let

$$R^iF(A) = h^i(F(I^\bullet)).$$

Now it is a well-known result that the category of modules over a ring has enough injectives.

Proposition 16.4. *Let $(X, \mathcal{O}_X)$ be a ringed space.*
Then the category of $\mathcal{O}_X$-modules has enough injectives.

Proof. Let $\mathcal{F}$ be a sheaf. For every $x \in X$ embed the stalk into an injective $\mathcal{O}_{X,x}$-module, $\mathcal{F}_x \longrightarrow I_x$. Let j denote the inclusion of $\{ x \}$ into X and let

$$\mathcal{I} = \prod_{x\in X} j_* I_x.$$

Now suppose that we have a sheaf $\mathcal{G}$ of $\mathcal{O}_X$-modules. Then

$$\begin{aligned}\mathbf{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{I}) &= \prod_{x\in X} \mathbf{Hom}_{\mathcal{O}_X}(\mathcal{G}, j_* I_x) \\ &= \prod_{x\in X} \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, I_x).\end{aligned}$$

In particular there is a natural map $\mathcal{F} \longrightarrow \mathcal{I}$, which is injective, as it is injective on stalks. Secondly the functor

$$\mathcal{G} \longrightarrow \mathbf{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{I}),$$

is the direct product over all $x \in X$ of the functor which sends $\mathcal{G}$ to its stalk, which is exact, followed by the functor,

$$\mathcal{G}_x \longrightarrow \mathbf{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, I_x),$$

which is exact, as I_x is an injective module. Thus

$$\mathcal{G} \longrightarrow \mathbf{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{I}),$$

is exact, which means that $\mathcal{I}$ is injective. □

Corollary 16.5. *Let X be a topological space.*
Then the category of sheaves on X has enough injectives.

Proof. The category of sheaves on X is equivalent to the category of $\mathcal{O}_X$-modules on the ringed space $(X, \mathbb{Z})$. □

If X is a topological space then we define $H^i(X, \mathcal{F})$ to be the right derived functor of

$$\mathcal{G} \longrightarrow \Gamma(X, \mathcal{G}).$$

Given an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0,$$

we get a long exact sequence of cohomology, so that we indeed fix the lack of exactness.

It is interesting to examine why we don't work with projective sheaves instead of injective sheaves. After all, projective modules are much easier to understand than injective modules. A module is projective if and only if it is direct summand of a free module, so any free module is projective.

However if X is a topological space there are almost never enough projectives in the category of sheaves on X. For example, suppose we have a topological space with the following property. There is a closed point $x \in X$ such that for any neighbourhood V of x in X there is a smaller connected open neighbourhood U of x, that is,

$$x \in U \subset V \subset X,$$

where $U \neq V$ is connected. Let $\mathcal{F} = \mathbb{Z}_{\{x\}}$ be the extension by zero of the constant sheaf $\mathbb{Z}$ on x, so that

$$\mathcal{F}(W) = \begin{cases} \mathbb{Z} & \text{if } x \in W \\ 0 & \text{otherwise.} \end{cases}$$

I claim that $\mathcal{F}$ is not the quotient of a projective sheaf. Suppose that

$$\mathcal{P} \longrightarrow \mathcal{F},$$

is a morphism of sheaves, where $\mathcal{P}$ is projective. Let V be any open neighbourhood of x. Pick

$$x \in U \subset V \subset X,$$

where $U \neq V$ is open and connected. Let $\mathcal{G} = \mathbb{Z}_U$ be the extension by zero of the locally constant sheaf $\mathbb{Z}$ on U, so that

$$\mathcal{G}(W) = \begin{cases} \mathbb{Z} & \text{if } W \subset U \\ 0 & \text{otherwise.} \end{cases}$$

As $\mathcal{P}$ is projective we have a commutative diagram

$$\begin{tikzcd} \mathcal{P} \arrow[r, dashed] \arrow[dr] & \mathcal{G} \arrow[d] \\ & \mathcal{F}. \end{tikzcd}$$

But $\mathcal{G}(V) = \mathbb{Z}_U(V) = 0$, so that the map $\mathcal{P}(V) \longrightarrow \mathcal{F}(V)$ is the zero map. But then the map on stalks is zero, so that the map $\mathcal{P} \longrightarrow \mathcal{F}$ is not surjective and $\mathcal{F}$ is not the quotient of a projective sheaf.

Note that if $(X, \mathcal{O}_X)$ is a ringed space then there are potentially two different ways to take the right derived functors of $\Gamma(X, \mathcal{F})$, if $\mathcal{F}$ is an $\mathcal{O}_X$-module. We could forget that X is a ringed space or we could work in the smaller category of $\mathcal{O}_X$-modules. We check that it does not matter in which category we work.

Definition 16.6. *Let $\mathcal{F}$ be a sheaf. We say that $\mathcal{F}$ is **flasque** if for every pair of open subsets $V \subset U \subset X$ the natural map*

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(V),$$

is surjective.

For any open subset $U \subset X$ let $\mathcal{O}_U$ be the extension by zero of the structure sheaf on U, $\mathcal{O}_X|_U$.

Lemma 16.7. *If $(X, \mathcal{O}_X)$ is a ringed space then every injective $\mathcal{O}_X$-module $\mathcal{I}$ is flasque.*

Proof. Let $V \subset U$ be open subsets. Then we have an inclusion $\mathcal{O}_V \longrightarrow \mathcal{O}_U$ of sheaves of $\mathcal{O}_X$-modules. As $\mathcal{I}$ is injective we get a surjection

$$\operatorname{Hom}(\mathcal{O}_U, \mathcal{I}) = \mathcal{I}(U) \longrightarrow \operatorname{Hom}(\mathcal{O}_V, \mathcal{I}) = \mathcal{I}(V). \qquad \square$$

Lemma 16.8. *If $\mathcal{F}$ is a flasque sheaf on a topological space X then*

$$H^i(X, \mathcal{F}) = 0,$$

for all $i > 0$.

Proof. Embed $\mathcal{F}$ into an injective sheaf and take the quotient to get a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{G} \longrightarrow 0.$$

As $\mathcal{I}$ is injective it is flasque and so $\mathcal{G}$ is flasque. As $\mathcal{F}$ is flasque there is an exact sequence

$$0 \longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{I}) \longrightarrow H^0(X, \mathcal{G}) \longrightarrow 0,$$

which taking the long exact sequence of cohomology, shows that

$$H^1(X, \mathcal{F}) = H^1(X, \mathcal{I}) = 0,$$

and

$$H^i(X, \mathcal{F}) = H^{i-1}(X, \mathcal{G}),$$

which is zero by induction on i. $\square$

Proposition 16.9. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then the derived functors of*

$$\mathcal{F} \longrightarrow \operatorname{Hom}(X, \mathcal{F}),$$

for either the category of sheaves of $\mathcal{O}_X$-modules or simply the category of sheaves on X to the category of abelian groups coincide.

Proof. Take an injective resolution of $\mathcal{F}$ in the category of $\mathcal{O}_X$-modules. Injective is flasque and flasque is acyclic, so this gives us a resolution by acylics in the category of sheaves on X and this is enough to calculate the right derived functors. □

Suppose that X is scheme over an affine scheme

$$X \longrightarrow S = \operatorname{Spec} A.$$

Now

$$H^0(S, \mathcal{O}_S) = A,$$

and all higher cohomology of any $\mathcal{O}_X$-module $\mathcal{F}$ is naturally an A-module.