21. Riemann-Roch

One of the most interesting applications of sheaf cohomology are very useful formulae for the number of global sections.

Definition 21.1. *Let $P(z) \in \mathbb{Q}[z]$ be a polynomial. We say that $P(z)$ is* ***numerical*** *if $P(n) \in \mathbb{Z}$ for any sufficiently large integer n.*

Lemma 21.2.

(1) *If $P(z)$ is a numerical polynomial then we may find integers $c_0, c_1, \dots, c_r$ such that*
$$P(z) = c_0\binom{z}{r} + c_1\binom{z}{r-1} + \dots + c_r.$$
In particular $P(n) \in \mathbb{Z}$ for every $n \in \mathbb{Z}$.

(2) *If $f \colon \mathbb{Z} \longrightarrow \mathbb{Z}$ is any function and there is a numerical polynomial $Q(z)$ such that $\Delta(f) = f(n+1) - f(n) = Q(n)$ for n sufficiently large then there is a numerical polynomial $P(z)$ such that $f(n) = P(n)$ for n sufficiently large.*

Proof. We prove (1) by induction on the degree r of P. Since
$$\binom{z}{r} = \frac{z(z-1)\cdots(z-r+1)}{r!} = \frac{z^r}{r!} + \dots,$$
is a polynomial of degree n, they form a basis for all polynomials and we may certainly find rationals $c_0, c_1, \dots, c_r$ such that
$$P(z) = c_0\binom{z}{r} + c_1\binom{z}{r-1} + \dots + c_r.$$
Note that
$$Q(z) = \Delta P(z) = P(z+1) - P(z) = c_0\binom{z}{r-1} + c_1\binom{z}{r-2} + \dots + c_{r-1},$$
is a numerical polynomial. By induction on the degree, $c_0, c_1, \dots, c_{r-1}$ are integers. It follows that c_r is an integer, as $P(n)$ is an integer for n large. This is (1).

For (2), suppose that
$$Q(z) = c_0\binom{z}{r} + c_1\binom{z}{r-1} + \dots + c_r,$$
for integers $c_0, c_1, \dots, c_r$. Let
$$P(z) = c_0\binom{z}{r+1} + c_1\binom{z}{r} + \dots + c_r\binom{z}{1}.$$

Then $\Delta P(z) = Q(z)$ so that $(f-P)(n)$ is a constant c_{r+1} for any n sufficiently large, so that $f(n) = P(n) + c_{r+1}$ for any n sufficiently large. $\square$

Theorem 21.3 (Asymptotic Riemann-Roch)**.** *Let X be a normal projective variety of dimension n and let $\mathcal{O}_X(1)$ be a very ample line bundle. Suppose that $X \subset \mathbb{P}^k$ has degree d.*

Then
$$h^0(X, \mathcal{O}_X(m)) = \frac{dm^n}{n!} + ...,$$
is a polynomial of degree n, for m large enough, with the given leading term.

Proof. First suppose that X is smooth. Let Y be a general hyperplane section. Then Y is smooth by Bertini. The trick is to compute $\chi(X, \mathcal{O}_X(m))$ by looking at the exact sequence
$$0 \longrightarrow \mathcal{O}_X(m-1) \longrightarrow \mathcal{O}_X(m) \longrightarrow \mathcal{O}_Y(m) \longrightarrow 0.$$
The Euler characteristic is additive so that
$$\chi(X, \mathcal{O}_X(m)) - \chi(X, \mathcal{O}_X(m-1)) = \chi(Y, \mathcal{O}_Y(m)).$$
(21.2) implies that $\chi(X, \mathcal{O}_X(m))$ is a polynomial of degree n, with the given leading term. Now apply Serre vanishing.

For the general case we need that if X is normal and Y is a general hyperplane section, then Y is a normal projective variety of degree d. Y is regular in codimension one by a Bertini type argument and one can check that Y is S_2. $\square$

It is fun to use similar arguments to prove special cases of Riemann-Roch.

Theorem 21.4 (Riemann-Roch for curves)**.** *Let C be a smooth projective curve of genus g and let D be a divisor of degree d.*
$$h^0(X, \mathcal{O}_C(D)) = d - g + 1 + h^0(C, \mathcal{O}_C(K_C - D)).$$

Proof. We first check that
$$\chi(C, \mathcal{O}_C(D)) = d - g + 1.$$
We may write
$$D = \sum a_i p_i.$$
We proceed by induction on $\sum |a_i|$. Let $p = p_1$. If $a_1 > 0$ then consider the short exact sequence
$$0 \longrightarrow \mathcal{O}_C(D-p) \longrightarrow \mathcal{O}_C(D) \longrightarrow \mathcal{O}_p \longrightarrow 0.$$

The Euler characteristic is additive, so that

$$\chi(C, \mathcal{O}_C(D)) = \chi(C, \mathcal{O}_C(D - p)) + 1.$$

The LHS is equal to $(d-1) - g + 1 + 1 = d - g + 1$ by induction. If $a_1 < 0$ then consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_C(D) \longrightarrow \mathcal{O}_C(D + p) \longrightarrow \mathcal{O}_p \longrightarrow 0.$$

The Euler characteristic is additive, so that

$$\chi(C, \mathcal{O}_C(D)) = \chi(C, \mathcal{O}_C(D + p)) - 1.$$

The RHS is equal to $d - g + 1 - 1 = (d-1) - g + 1$ by induction.

So we are reduced to the case when $d = 0$. Note that

$$h^1(C, \mathcal{O}_C(D)) = h^0(C, \mathcal{O}_C(K_C - D)),$$

by Serre duality. In particular

$$\chi(C, \mathcal{O}_C) = 1 - g,$$

which completes the induction. $\square$

Theorem 21.5 (Riemann-Roch for surfaces)**.** *Let S be a smooth projective surface of irregularity q and geometric genus p_g over an algebraically closed field of characteristic zero. Let D be a divisor on S.*

$$\chi(S, \mathcal{O}_S(D)) = \frac{D^2}{2} - \frac{K_S \cdot D}{2} + 1 - q + p_g.$$

Proof. Pick a very ample divisor H such that $H + D$ is very ample. Let C and Σ be general elements of $|H|$ and $|H + D|$. Then C and Σ are smooth. There are two exact sequences

$$0 \longrightarrow \mathcal{O}_S(D) \longrightarrow \mathcal{O}_S(D + H) \longrightarrow \mathcal{O}_C(D + H) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(D + H) \longrightarrow \mathcal{O}_\Sigma(D + H) \longrightarrow 0.$$

As the Euler characteristic is additive we have

$$\begin{aligned}\chi(S, \mathcal{O}_S(D + H)) &= \chi(S, \mathcal{O}_S(D)) + \chi(C, \mathcal{O}_C(D + H)) \\ \chi(S, \mathcal{O}_S(D + H)) &= \chi(S, \mathcal{O}_S) + \chi(\Sigma, \mathcal{O}_\Sigma(D + H)).\end{aligned}$$

Subtracting we get

$$\chi(S, \mathcal{O}_S(D)) - \chi(S, \mathcal{O}_S) = \chi(\Sigma, \mathcal{O}_\Sigma(D + H)) - \chi(C, \mathcal{O}_C(D + H)).$$

Now

$$\begin{aligned}\chi(\Sigma, \mathcal{O}_\Sigma(D + H)) &= (D + H) \cdot \Sigma - \deg K_\Sigma/2 \\ \chi(C, \mathcal{O}_C(D + H)) &= (D + H) \cdot C - \deg K_C/2,\end{aligned}$$

applying Riemann-Roch for curves to both C and Σ. We have

$$(D+H)\cdot\Sigma = (D+H)\cdot H + (D+H)\cdot D,$$

and by adjunction

$$K_\Sigma = (K_S+\Sigma)\cdot\Sigma \qquad \text{and} \qquad K_C = (K_S+C)\cdot C.$$

So putting all of this together we get

$$\begin{aligned}\chi(S,\mathcal{O}_S(D)) - \chi(S,\mathcal{O}_S) &= (D+H)\cdot D + \frac{1}{2}((K_S+C)\cdot C - (K_S+\Sigma)\cdot\Sigma) \\ &= (D+H)\cdot D + \frac{1}{2}K_S\cdot(C-\Sigma) + \frac{1}{2}(H\cdot H - (H+D)\cdot(H+D)) \\ &= \frac{D\cdot D}{2} - \frac{1}{2}K_S\cdot D.\end{aligned}$$

We have

$$c = \chi(S,\mathcal{O}_S) = h^0(S,\mathcal{O}_S) - h^1(S,\mathcal{O}_S) + h^2(S,\mathcal{O}_S) = 1 - q + p_g.$$

Here we used the highly non-trivial fact that

$$h^1(S,\mathcal{O}_S) = h^0(S,\Omega^1_S) = q,$$

from Hodge theory and Serre duality

$$h^2(S,\mathcal{O}_S) = h^0(S,\omega_S) = p_g. \qquad \square$$

Remark 21.6. *One can turn Riemann-Roch for surfaces around and use the arguments in the proof of* (21.5) *to prove basic properties of the intersection number.*