22. -1-curves

We will need a little bit of intersection theory.

Given any variety X we can define cycles of any dimension on X. A **cycle** α is a formal linear combination of closed subvarieties, $\sum n_V V$. If V all have the same dimension k then we say α is a k**-cycle**. Two k-cycles α and β are **rationally equivalent** if there is a $k+1$ dimensional subvariety W which contains the support of α and β and α and β are linearly equivalent divisors on the normalisation of W. If X has dimension n then an $(n-1)$-cycle is the same as a Weil divisor.

Note that we can pullback Cartier divisors. We can also pushforward Weil divisors, or more generally cycles. If $f\colon X \longrightarrow Y$ is a proper morphism and V is an irreducible closed subvariety with image W then

$$f_*V = \begin{cases} dW & \text{if } f|_V \colon V \longrightarrow W \text{ is generically finite of degree } d \\ 0 & \text{otherwise.} \end{cases}$$

In other words if the image of V is lower dimensional then $f_*V = 0$. If the image W of V has the same dimension then $f_*V = dW$ where d is the degree of V over W. We extend the pushforward by linearity to all cycles.

Note that we can intersect a cycle α with a Cartier divisor D, to get a cycle $\alpha \cdot D$. By linearity we may assume that $\alpha = V$ is a closed irreducible subvariety. In this case we can define a linear equivalence of Cartier divisors on V. If the support of D does not contain then simply restrict the equations of D to V. If the support of D does contain V then restrict the invertible sheaf $\mathcal{O}_X(D)$ to V to get an invertible sheaf on V. An invertible sheaf is the same as a linear equivalence class of Cartier divisors. Now pushforward the corresponding Weil divisor, via the natural inclusion $V \longrightarrow X$ to get a cycle on X.

Now pushforward is not a ring homomorphism, but it is almost is:

Theorem 22.1 (Push-pull)**.** *Let $f\colon X \longrightarrow Y$ be a proper morphism of varieties. Let α be a cycle on X and let D be a Cartier divisor on Y.*

Then
$$f_*(\alpha \cdot f^*D) = f_*\alpha \cdot D.$$

0-cycles are formal sums of points $\sum n_p p$. The degree is the sum $\sum n_p$. Note that two rationally equivalent 0-cycles have the same degree.

If X is a smooth projective variety over $\mathbb{C}$ then we can associate to any cycle α a class in homology. As usual, by linearity it is enough to do this for irreducible subvarieties V. Take a simplicial decomposition of X which induces a simplicial decomposition of V. Then V defines a class $[V] \in H_*(X, \mathbb{Z})$. A divisor D determines a class in cohomology $[D] \in H^2(X, \mathbb{Z})$. We can pair this with a homology classes. This is compatible with the algebraic intersection product

$$[D \cdot \alpha] = [D] \cap [\alpha] \in H_*(X, \mathbb{Z}).$$

Note also that there is a topological push-pull formula.

Now let us consider what happens for surfaces. 1-cycles, or Weil divisors, are nothing more than formal sums of curves. If we intersect a Weil divisor with a Cartier divisor, we will get a rational equivalence class of 0-cycles. The intersection number is just the degree of the 0-cycles. From now on, the intersection product will denote the degree.

We can compute the degree locally.

Definition 22.2. *Let S be a smooth surface and let p be a point of S. Let D_1 and D_2 be two Cartier divisors on S.*

First suppose that $D_1 = C_1$ and $D_2 = C_2$ are prime divisors. The ***local intersection number*** *of D_1 and D_2 at p,*

$$i_p(D_1, D_2) = \dim_k \mathcal{O}_{S,p}/\langle f_1, f_2 \rangle,$$

where f_1 and f_2 are local generators of the ideals of C_1 and C_2.

Now extend this by linearity to any two divisors with no common components.

It is interesting to check that the local intersection number coincides with geometric intuition.

Example 22.3. *Let $S = \mathbb{A}^2$, let C_1 be the x-axis $= 0$ and let C_2 be the conic $y = x^2$. Then C_1 and C_2 are tangent. The local intersection number is the dimension of the k-vector space*

$$\frac{k[x, y]}{\langle y, y - x^2 \rangle} = \frac{k[x]}{\langle x^2 \rangle} = k\langle 1, x \rangle$$

which is two, as expected.

Proposition 22.4. *If S is a projective surface and D_1 and D_2 are two divisors with no common components*

$$D_1 \cdot D_2 = \sum_p i_p(D_1, D_2).$$

Here the sum is over all points p in the interesection.

Theorem 22.5 (Bézout's Theorem)**.** *Let C and D be two curves defined by homogenous polynomials of degrees d and e. Suppose that $C \cap D$ does not contain a curve.*

Then $|C \cap D|$ is at most de, with equality if and only if the intersection of the two tangent spaces at $p \in C \cap D$ is equal to p.

Proof. $C \sim dL$ and $D \sim eL$, where L is a line. In this case

$$|C \cap D| \leq \sum_p i_p(C, D) = C \cdot D = (dL) \cdot (eL) = (de)L^2 = de. \qquad \square$$

Definition 22.6. *Let $C \subset S$ be a curve in a smooth surface. Let $p \in S$ be a point of S.*

The ***multiplicity of*** C ***at*** $p \in S$ *is the largest μ such that $\mathcal{I}_p \subset \mathfrak{m}^\mu$ where $\mathfrak{m}$ is the maximal ideal of S at p in $\mathcal{O}_{S,p}$ and $\mathcal{I}$ is the ideal sheaf of C in S.*

Note that $\mathcal{I} = \langle f \rangle$, so we just want the largest μ such that $f \in \mathfrak{m}^\mu$. If we work over $\mathbb{C}$, then we can choose coordinates x and y. In this case $\mathfrak{m} = \langle x, y \rangle$ and f is a power series in x and y. If we expand f in powers of x and y,

$$f(x, y) = f_0 + f_1 + f_2 + \dots$$

where f_i is homogenous of degree i then the multiplicity μ is the smallest integer such that $f_\mu \neq 0$.

Lemma 22.7. *Let $C \subset S$ be a curve in a smooth surface. Let $p \in S$ be a point of S and let $\pi \colon T \longrightarrow S$ be the blow up of S at p. Let $\tilde{C}$ be the strict transform of C. Then*

$$\pi^* C = \tilde{C} + \mu E.$$

Proof. Pick coordinates so that $y = 0$ is not tangent to any branch of C. Then $T \subset S \times \mathbb{P}^1$ and local coordinates on T are given by (x, t), where $y = tx$. In this case

$$f(x, y) = f_\mu(x, xt) + f_{\mu+1}(x, xt) = x^\mu (f_\mu(1, t) + x f_\mu(1, t) + \dots).$$

As $x = 0$ is the equation of E the result is clear. $\square$

Proposition 22.8. *Let S be a smooth surface, let $p \in S$ be a point of S and let $\pi \colon T \longrightarrow S$ be the blow up of S at p, with exceptional divisor E.*

Then $E^2 = -1$.

Proof. We give two proofs of this result.

Here is the first. Note that this is a local computation. So we might as well assume that $S = \mathbb{P}^2$. Pick a line L passing through p. Then L is a Cartier divisor on S. We have

$$\pi^* L = M + E,$$

where M is the strict transform of L.

Let L_1 and L_2 be two general lines through p. Then

$$L^2 = L_1 \cdot L_2 = 1,$$

that is, two lines meet in one point. Let M_1 and M_2 be the strict transforms of L_1 and L_2. Then M_1 and M_2 don't meet, by definition of the blow up. Thus

$$M^2 = M_1 \cdot M_2 = 0.$$

To practice using push-pull, let us calculate $E \cdot \pi^* L$. By push-pull,

$$E \cdot \pi^* L = \pi_*(E \cdot \pi^* L) = \pi_* E \cdot L = 0.$$

Thus

$$1 = M \cdot \pi^* L = M \cdot (M + E) = M \cdot M + M \cdot E = 1,$$

which is consistent.

Now let us calculate $E \cdot \pi^* L$. By push-pull this is

$$\pi_*(E \cdot \pi^* L) = \pi_* E \cdot L = 0,$$

since $\pi_* E = 0$. On the other hand,

$$E \cdot \pi^* L = E \cdot (M + E) = E \cdot M + E^2 = 1 + E^2.$$

Thus $E^2 = -1$.

Here is the second method. The ideal sheaf of E in T is given by $\mathcal{O}_T(-E)$. By definition of the blow up, this restricts to $\mathcal{O}_E(1) = \mathcal{O}_{\mathbb{P}^1}(1)$. Thus $\mathcal{O}_T(E)$ restricts to $\mathcal{O}_E(-1)$, so that $E|_E$ has degree -1. □

Definition-Lemma 22.9. *Let S be a smooth surface and let $C \subset S$ be a proper irreducible curve.*

Any two of the following three properties implies the third:

(1) $C \simeq \mathbb{P}^1$.
(2) $C^2 = -1$.
(3) $K_S \cdot C = -1$.

In this case we call E a -1-curve.

Proof. By adjunction

$$(K_S + C)|_C = K_C.$$

Thus

$$2g - 2 = \deg K_C = (K_S + C) \cdot C.$$

Note that $C \simeq \mathbb{P}^1$ if and only if $g = 0$. The result is then clear. □

Lemma 22.10. *Let $\pi\colon T \longrightarrow S$ be the blow up of a smooth point of a smooth surface. Let E be the exceptional divisor.*

Then

$$K_T = \pi^* K_S + E.$$

Proof. Note that π is an isomorphism outside p, so that

$$K_T = \pi^* K_S + aE,$$

for some integer a. It suffices to check that $a = 1$; we will give two proofs of this result.

Here is the first. We have already seen that $E \simeq \mathbb{P}^1$ and $E^2 = -1$. So $K_T \cdot E = -1$ by (22.9). On the other hand,

$$-1 = K_T \cdot E = (\pi^* K_S + aE) \cdot E = K_S \pi_* E + aE^2 = -a.$$

Thus $a = 1$.

The second is by direct computation. Let (x, y) be local coordinates on S. Then

$$\omega = \mathrm{d}x \wedge \mathrm{d}y,$$

is a meromorphic differential with no poles or zeroes in a neighbourhood of p. Local coordinates upstairs are (x, t), where $y = xt$.

$$\begin{aligned} \pi^*\omega &= \mathrm{d}x \wedge \mathrm{d}(xt) \\ &= \mathrm{d}x \wedge (t\mathrm{d}x + x\mathrm{d}(t) \\ &= t\mathrm{d}x \wedge \mathrm{d}x + x\mathrm{d}x \wedge \mathrm{d}(t) \\ &= x\mathrm{d}x \wedge \mathrm{d}t. \end{aligned}$$

Thus the pullback of a meromorphic differential from S always has a simple zero along E. □

Lemma 22.11. *Let $\pi\colon T \longrightarrow S$ be the blow up of a smooth point of a smooth surface.*

Then

$$K_T^2 = K_S^2 - 1.$$

Proof.

$$K_T^2 = K_T \cdot (\pi^* K_S + E) = K_T \cdot \pi^* K_S + K_T \cdot E = K_S^2 - 1. \qquad \square$$