## 4. Sard's Theorem

A basic result in the theory of $C^\infty$-maps is Sard's Theorem, which states that the set of points where a map is singular is a subset of measure zero (of the base). Since any holomorphic map between complex manifolds is automatically $C^\infty$, and the derivative of a polynomial is the same as the derivative as a holomorphic function, it follows that any morphism between varieties, over $\mathbb{C}$, is smooth over an open subset. In fact by the Lefschetz principle, this result extends to any variety over $\mathbb{C}$.

**Theorem 4.1.** *Let $f\colon X \longrightarrow Y$ be a morphism of varieties over a field of characteristic zero.*

*Then there is a dense open subset $U$ of $Y$ such that if $q \in U$ and $p \in f^{-1}(q) \cap X_{sm}$ then the differential $df_p\colon T_pX \longrightarrow T_qY$ is surjective. Further, if $X$ is smooth, then the fibres $f^{-1}(q)$ are smooth if $q \in U$.*

Let us recall the Lefschetz principle. First recall the notion of a first order theory of logic. Basically this means that one describes a theory of mathematics using a theory based on predicate calculus. For example, the following is a true statement from the first order theory of number theory,

$$\forall n \forall x \forall y \forall z \quad n \geq 3 \implies x^n + y^n \neq z^n.$$

One basic and desirable property of a first order theory of logic is that it is complete. In other words every possible statement (meaning anything that is well-formed) can be either proved or disproved. It is a very well-known result that the first order theory of number theory is not complete (Gödel's Incompleteness Theorem). What is perhaps more surprising is that there are interesting theories which are complete.

**Theorem 4.2.** *The first order logic of algebraically closed fields of characteristic zero is complete.*

Notice that a typical statement of the first order logic of fields is that a system of polynomial equations does or does not have solution. Since most statements in algebraic geometry turn on whether or not a system of polynomial equations have a solution, the following result is very useful.

**Principle 4.3** (Lefschetz Principle)**.** *Every statement in the first order logic of algebraically closed fields of characteristic zero, which is true over $\mathbb{C}$, is in fact true over any algebraically closed field of characteristic.*

In fact this principle is immediate from (4.2). Suppose that $p$ is a statement in the first order logic of algebraically closed fields of characteristic zero. By completeness, we can either prove $p$ or not $p$. Since $p$ holds over the complex numbers, there is no way we can prove not $p$. Therefore there must be a proof of $p$. But this proof is valid over any field of characteristic zero, so $p$ holds over any algebraically closed field of characteristic zero.

A typical application of the Lefschetz principle is (4.1). By Sard's Theorem, we know that (4.1) holds over $\mathbb{C}$. On the other hand, given a variety $X$ over an algebraically closed field, of characteristic zero, whether or not (4.1) holds for $X$, can be reformulated in the first order logic of algebraically closed fields of characteristic zero. Therefore, by the Lefschetz principle, (4.1) is true for $X$, so that (4.1) is true over an algebraically closed field of characteristic zero.

One way to think of the Lefschetz principle is as follows. Take a variety $X$ defined over an algebraically closed field $L$ of characteristic zero. Realise $X \subset \mathbb{P}^n$. Then $X$ is defined by finitely many polynomials $F_1, F_2, \dots, F_k$. The coefficients of these polynomials define a finitely generated extension $L/K/\mathbb{Q}$. Therefore we can find a scheme $X'$ over $K$ such that $X$ is obtained from $X$ by the base change $\operatorname{Spec} L \longrightarrow \operatorname{Spec} K$. On the other hand, $K$ can be embedded into $\mathbb{C}$. The base change $\operatorname{Spec} \mathbb{C} \longrightarrow \operatorname{Spec} K$ gives us a variety $Y$ over $\mathbb{C}$. Most of the time, properties of $X$ follow from properties of $Y$.

Perhaps even more interesting is that (4.1) fails in characteristic $p$. Let $f\colon \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ be the morphism $t \longrightarrow t^p$. If we fix $s$, then the equation

$$x^p - s$$

is purely inseparable, that is, has only one root. Thus $f$ is a bijection. However, $df$ is the zero map, since $dz^p = pz^{p-1}dz = 0$. Thus $df_p$ is nowhere surjective. Note that the fibres of this map, as schemes, are isomorphic to zero dimensional schemes of length $p$.