6. Tangent lines to plane curves

Question 6.1. *Let C be a curve in $\mathbb{P}^2$ and let $p \in \mathbb{P}^2$.*

How many tangent lines does p lie on?

The first thing that we will need is a natty way to describe the projective tangent space to a variety.

Definition 6.2. *Let $X \subset \mathbb{P}^n$.*

The ***projective tangent space to** X **at*** *p is the closure of the affine tangent space.*

(Note the difference between the projective tangent space and the projectivisation of the tangent space.) In other words the projective tangent space has the same dimension as the affine tangent space and is obtained by adding the suitable points at infinity. Suppose that the curve is defined by the polynomial $F(X, Y, Z)$. Then the tangent line to C at p, is

$$\left.\frac{\partial F}{\partial X}\right|_p X + \left.\frac{\partial F}{\partial Y}\right|_p Y + \left.\frac{\partial F}{\partial Z}\right|_p Z.$$

Of course it suffices to check that we get the right answer on an affine piece.

Lemma 6.3. *Let F be a homogeneous polynomial of degree d in $X_0, X_1, \dots, X_n$. Then*

$$dF = \sum X_i \frac{\partial F}{\partial X_i}$$

Proof. Both sides are linear in F. Thus it suffices to prove this for a monomial of degree d, when the result is clear. $\square$

It follows then that the tangent line above does indeed pass through p. The rest is easy.

Finally we will need Bézout's Theorem.

Theorem 6.4 (Bézout's Theorem)**.** *Let C and D be two curves defined by homogenous polynomials of degrees d and e. Suppose that $C \cap D$ does not contain a curve.*

Then $|C \cap D|$ is at most de, with equality iff the intersection of the two tangent spaces at $p \in C \cap D$ is equal to p.

We are now ready to answer (6.1).

Lemma 6.5. *Let $C \subset \mathbb{P}^n$ be a curve in $\mathbb{P}^2$ and let $p \in \mathbb{P}^2$ be a general point.*

Then p lies on $d(d-1)$ tangent lines.

Proof. Fix $p = [a : b : c]$ and let D be the curve defined by

$$G = a\frac{\partial F}{\partial X} + b\frac{\partial F}{\partial Y} + c\frac{\partial F}{\partial Z}.$$

Then G is a polynomial of degree $d-1$. Consider a point q where C intersects D. Then the tangent line to C at q is given by

$$\left.\frac{\partial F}{\partial X}\right|_q X + \left.\frac{\partial F}{\partial Y}\right|_q Y + \left.\frac{\partial F}{\partial Z}\right|_q Z.$$

But then since p satisfies this equation, as q lies on D, it follows that p lies on the tangent line of C at q. Similarly it is easy to check the converse, that if p lies on the tangent line to C at q, then q is an intersection point of C and D.

Now apply Bézout's Theorem. $\square$

There is an interesting way to look at all of this. In fact one may generalise the result above to the case of curves with nodes. Note that if you take a curve in $\mathbb{P}^3$ and take a general projection down to $\mathbb{P}^2$, then you get a nodal curve. Indeed it is easy to pick the point of projection not on a tangent line, since the space of tangent lines obviously sweeps out a surface; it is a little more involved to show that the space of three secant lines is a proper subvariety. (6.5) was then generalised to this case and it was shown that if δ is the number of nodes, then the number

$$\frac{(d-1(d-2)}{2} - \delta$$

is an invariant of the curve.

Here is another way to look at this. Suppose that we project our curve down to $\mathbb{P}^1$ from a point. Then we get a finite cover of $\mathbb{P}^1$, with d points in the general fibre. Lines tangent to C passing through p then count the number of branch points, that is, the number of points in the base where the fibre has fewer than d points. Since this tangent line is only tangent to p and is simply tangent (that is, there are no flex points) there are $d-1$ points in this fibre, and the ramification point corresponding to the branch point is where two sheets come together.

The modern approach to this invariant is quite different. If we are over the complex numbers $\mathbb{C}$, changing perspective, we may view the curve C as a Riemann surface covering another Riemann surface D. Now the basic topological invariant of a compact oriented Riemann surface is it's genus. In these terms there is a simple formula that connects the genus of C and B, in terms of the ramification data, known as Riemann-Hurwitz,

$$2g - 2 = d(2h-2) + b,$$

where g is the genus of C, h the genus of B, d the order of the cover and b the contribution from the ramification points. Indeed if locally on C, the map is given as $z \longrightarrow z^e$ so that e sheets come together, the contribution is $e-1$.

In our case, $B = \mathbb{P}^1$ which is of genus 0, for each branch point, we have simple ramification, so that $e = 2$ and the contribution is one, making a total $b = d(d-1)$. Thus

$$2g - 2 = -2d + d(d-1).$$

Solving for g we get

$$g = \frac{(d-1)(d-2)}{2}.$$

Note that if $d \leq 2$, then we get $g = 0$ as expected (that is $C \simeq \mathbb{P}^1$) and if $d = 3$ then we get an elliptic curve.