8. Linear systems

Definition 8.1. *Let $\mathcal{L}$ be an invertible sheaf on a smooth projective variety over an algebraically closed field. Let $s \in H^0(X, \mathcal{L})$. The divisor (s) of zeroes of s is defined as follows. By assumption we may cover X by open subsets U_i over which we may identify $s|_{U_i}$ with $f_i \in \mathcal{O}_{U_i}$. The defines a Cartier divisor $\{(U_i, f_i)\}$.*

It is a simple matter to check that the Cartier divisor does not depend on our choice of trivialisations. Note that as X is smooth the Cartier divisor may safely be identified with the corresponding Weil divisor.

Lemma 8.2. *Let X be a smooth projective variety over an algebraically closed field. Let D_0 be a divisor and let $\mathcal{L} = \mathcal{O}_X(D_0)$.*

(1) If $s \in H^0(X, \mathcal{L})$, $s \neq 0$ then $(s) \sim D_0$.
(2) If $D \geq 0$ and $D \sim D_0$ then there is a global section $s \in H^0(X, \mathcal{L})$ such that $D = (s)$.
(3) If $s_i \in H^0(X, \mathcal{L})$, $i = 1$ and 2, are two global sections then $(s_1) = (s_2)$ if and only if $s_2 = \lambda s_1$ where $\lambda \in k^*$.

Proof. As $\mathcal{O}_X(D_0) \subset \mathcal{K}$, the section s corresponds to a rational function f. If D_0 is the Cartier divisor $\{(U_i, f_i)\}$ then $\mathcal{O}_X(D_0)$ is locally generated by f_i^{-1} so that multiplication by f_i induces an isomorphism with $\mathcal{O}_{U_i}$. D is then locally defined by ff_i. But then

$$D = D_0 + (f).$$

Hence (1).

Now suppose that $D > 0$ and $D = D_0 + (f)$. Then $(f) \geq -D_0$. Hence

$$f \in H^0(X, \mathcal{O}_X(D_0)) \subset H^0(X, \mathcal{K}) = K(X),$$

and the divisor of zeroes of f is D. This is (2).

Now suppose that $(s_1) = (s_2)$. Then

$$D_0 + (f_1) = (s_1) = (s_2) = D_0 + (f_2).$$

Cancelling, we get that $(f_1) = (f_2)$ and the rational function f_1/f_2 has no zeroes nor poles. Since X is a projective variety, $f_1/f_2 = \lambda$, a constant. □

Definition 8.3. *Let D_0 be a divisor. The* ***complete linear system*** *associated to D_0 is the set*

$$|D_0| = \{\, D \in \operatorname{Div}(X) \mid D \geq 0, D \sim D_0 \,\}.$$

We have seen that

$$|D| = \mathbb{P}(H^0(X, \mathcal{O}_X(D_0))).$$

Thus $|D|$ is naturally a projective space.

Definition 8.4. *A* ***linear system*** *is any linear subspace of a complete linear system $|D_0|$.*

In other words, a linear system corresponds to a linear subspace, $V \subset H^0(X, \mathcal{O}_X(D_0))$. We will then write

$$|V| = \{ D \in |D_0| \mid D = (s), s \in V \} \simeq \mathbb{P}(V) \subset \mathbb{P}(H^0(X, \mathcal{O}_X(D_0))).$$

Definition 8.5. *Let $|V|$ be a linear system. The* ***base locus*** *of $|V|$ is the intersection of the elements of $|V|$.*

Lemma 8.6. *Let X be a smooth projective variety over an algebraically closed field, and let $|V| \subset |D_0|$ be a linear system.*

V generates $\mathcal{O}_X(D_0)$ if and only if $|V|$ is base point free.

Proof. If V generates $\mathcal{O}_X(D_0)$ then for every point $x \in X$ we may find an element $\sigma \in V$ such that $\sigma(x) \neq 0$. But then $D = (\sigma)$ does not contain x, and so the base locus is empty.

Conversely suppose that the base locus is empty. The locus where V does not generate $\mathcal{O}_X(D_0)$ is a closed subset Z of X. Pick $x \in Z$ a closed point. By assumption we may find $D \in |V|$ such that $x \notin D$. But then if $D = (\sigma)$, $\sigma(x) \neq 0$ and σ generates the stalk $\mathcal{L}_x$, a contradiction. Thus Z is empty and $\mathcal{O}_X(D_0)$ is globally generated. $\square$

Example 8.7. *Consider $\mathcal{O}_{\mathbb{P}^1}(4)$. The complete linear system $|4p|$ defines a morphism into $\mathbb{P}^4$, where $p = [0 : 1]$ and $q = [1 : 0]$, given by $\mathbb{P}^1 \longrightarrow \mathbb{P}^4$, $[S : T] \longrightarrow [S^4 : ST^3 : S^2T^2 : ST^3 : T^4]$. If we project from $[0 : 0 : 1 : 0 : 0]$ we will get a morphism into $\mathbb{P}^3$, $[S : T] \longrightarrow [S^4 : ST^3 : ST^3 : T^4]$. This corresponds to the sublinear system spanned by $4p$, $3p + q$, $p + 3q$, $4q$.*

Consider $\mathcal{O}_{\mathbb{P}^2}(2)$ and the corresponding complete linear system. The map associated to this linear system is the Veronese embedding $\mathbb{P}^2 \longrightarrow \mathbb{P}^5$, $[X : Y : Z] \longrightarrow [X^2 : Y^2 : Z^2 : YZ : XZ : XY]$.

Note also the notion of separating points and tangent directions becomes a little clearer in this more geometric setting. Separating points means that given x and $y \in X$, we can find $D \in |V|$ such that $x \in D$ and $y \notin D$. Separating tangent vectors means that given any irreducible length two zero dimensional scheme z, with support x, we can find $D \in |V|$ such that $x \in D$ but z is not contained in D. In fact the condition about separating tangent vectors is really the limiting case of separating points.

Thinking in terms of linear systems also presents an inductive approach to proving global generation. Suppose that we consider the complete linear system $|D|$. Suppose that we can find $Y \in |D|$. Then the base locus of $|D|$ is supported on Y. On the other hand suppose that $\mathcal{I}$ is the ideal sheaf of Y in X. Then there is an exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

As X is smooth D is Cartier and $\mathcal{O}_X(D)$ is an invertible sheaf. Tensoring by locally free preserves exactness, so there are short exact sequences,

$$0 \longrightarrow \mathcal{I}(mD) \longrightarrow \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_Y(mD) \longrightarrow 0.$$

Taking global sections, we get

$$0 \longrightarrow H^0(X, \mathcal{I}(mD) \longrightarrow H^0(X, \mathcal{O}_X(mD) \longrightarrow H^0(Y, \mathcal{O}_Y(mD).$$

At the level of linear systems there is therefore a linear map

$$|D| \longrightarrow |D|_Y|.$$

Consider another application of the ideas behind this section. Consider the problem of parametrising subvarieties or subschemes X of projective space $\mathbb{P}^r_k$. Any subscheme is determined by the homogeneous ideal $I(X)$ of polynomials vanishing on X. As in the case of zero dimensional schemes, we would like to reduce to the data of a vector subspace of fixed dimension in a fixed vector space. The obvious thing to consider is polynomials of degree d and the vector subspace of polynomials of polynomials of degree d vanishing on X. But how large should we take d to be?

The first observation is that if $\mathcal{I}$ is the ideal sheaf of X in $\mathbb{P}^r_k$ then

$$I_d = H^0(\mathbb{P}^r_k, \mathcal{I}(d)),$$

where $\mathcal{I}(d)$ is the Serre twist. To say that I_d determines X, is essentially equivalent to saying that $\mathcal{I}(d)$ is globally generated. Fixing some data about X (in the case of zero dimensional schemes this would be the length) we would then like a positive integer d_0 such that if $d \geq d_0$ then two things are true:

- $\mathcal{I}(d)$ is globally generated.
- $h^0(\mathbb{P}^r_k, \mathcal{I}(d))$, the dimension of the space of global sections, is independent of X.

Now there is a short exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{\mathbb{P}^r_k} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Twisting by d, we get

$$0 \longrightarrow \mathcal{I}(d) \longrightarrow \mathcal{O}_{\mathbb{P}^r_k}(d) \longrightarrow \mathcal{O}_X(d) \longrightarrow 0.$$

Taking global sections gives another exact sequence.

$$0 \longrightarrow H^0(\mathbb{P}^r_k, \mathcal{I}(d)) \longrightarrow H^0(\mathbb{P}^r_k, \mathcal{O}_{\mathbb{P}^r_k}(d)) \longrightarrow H^0(X, \mathcal{O}_X(d)).$$

Again, it would be really nice if this exact sequence were exact on the right. Then global generation of $\mathcal{I}(d)$ would be reduced to global generation of $\mathcal{O}_X(d)$ and one could read of $h^0(\mathbb{P}^r_k, \mathcal{I}(d))$ from $h^0(X, \mathcal{O}_X(d))$.