MODEL ANSWERS TO HWK #3

1. (i) Call a line **standard** if it is either horizontal or vertical.
It is expedient to prove an even stronger result. We prove that if $f\colon U \longrightarrow \mathbb{C}$ is any function, where U is the complement of finitely many standard lines, which restricts to a polynomial on any standard line contained in U, then f is a polynomial. We will be somewhat sloppy and say that a standard line is contained in U if it is not one of the deleted lines (strictly speaking, only the line minus finitely many points lies in U).
Note that if $V \subset U$ is obtained from U by deleting finitely many more standard lines and $f|_V$ is a polynomial, then f is a polynomial. Indeed $f|_V$ extends to a polynomial function $g\colon U \longrightarrow \mathbb{C}$. If l is a line in U then $f|_l$ and $g|_l$ agree on an open subset of the line and so are equal. But then $f = g$.
Let d be the smallest positive integer such that there are uncountably many real numbers r such that the restriction of f to the vertical line $x = r$ is a polynomial of degree at most d and there are uncountably many real numbers s such that the restriction of f to the horizontal line $y = s$ is a polynomial of degree at most d.
We proceed by induction on d. Suppose that $d < 0$, so that $f(x, y)$ restricts to the zero function on infinitely many horizontal and infinitely many vertical lines. If l is any standard line contained in U then the restriction of f to l is a polynomial with infinitely many zeroes, so that f must be the zero function, which is represented by the zero polynomial.
Suppose that $d \geq 0$. Note that the change of coordinates $x \longrightarrow x - a$ does not change the property that U is the complement of finitely many standard lines, that f restricted to any standard line is a polynomial and it also does not change the value of d. So we might as well assume that the x-axis is contained in U and $f(x, 0)$ is a polynomial of degree at most d. Let $g(x, y) = f(x, y) - f(x, 0)$. Then the restriction of $g(x, y)$ to every vertical line is a polynomial in y which vanishes at the origin. Let $V \subset U$ be the set obtained by deleting the line $y = 0$. Let

$$h\colon V \longrightarrow \mathbb{C},$$

be the function $h(x, y) = f(x, y)/y$. Then V is obtained from $\mathbb{C}^2$ by deleting finitely many standard lines, $h(x, y)$ is a function which when restricted to any standard line in V is a polynomial, which has degree

at most $d-1$ on uncountably many standard lines. By induction $h(x, y)$ is a polynomial function. It follows that $f(x, y) = yh(x, y) + f(x, 0)$ is a polynomial function on V, whence on U. Thus $P(\mathbb{C})$ is true.
(ii) Enumerate, $c_1, c_2, \ldots$ the points of $\overline{\mathbb{Q}}$ and let $h_n(y)$ (respectively $v_n(x)$) be the polynomial which vanishes on the first n horizontal (respectively vertical) lines. Let

$$f(x, y) = \sum_{i=0}^{\infty} h_i(y)v_i(x).$$

It is clear that $f(x, y)$ is not a polynomial. But suppose we pick a horizontal line, given by, $y = b$. Then $b = c_n$ for some n and so

$$f(x, b) = \sum_{i \leq n} h_i(b)v_i(x),$$

so that $f(x, b)$ is a polynomial. By symmetry the restriction of $f(x, y)$ to any vertical line is a polynomial. So $P(\overline{\mathbb{Q}})$ fails.
(iii) Clear, from (i) and (ii) and the Lefschetz principle.
2. (i) Let $\phi \in V$, $\psi \in V$ and let $\lambda \in k$. Then

$$\phi \colon z \longrightarrow X,$$

is a morphism of schemes over k, such that the unique point of z goes to x. But then ϕ corresponds to a morphism of local rings over k,

$$f \colon \mathcal{O}_{X,x} \longrightarrow \frac{k[\epsilon]}{\langle \epsilon^2 \rangle}.$$

Similarly suppose that ψ corresponds to g. Note that the function

$$m_\lambda \colon \frac{k[\epsilon]}{\langle \epsilon^2 \rangle} \longrightarrow \frac{k[\epsilon]}{\langle \epsilon^2 \rangle} \qquad \text{given by} \qquad a + b\epsilon \longrightarrow a + \lambda b\epsilon,$$

is a morphism of local rings, which is an isomorphism if and only if $\lambda \neq 0$. Let $\lambda\phi$ be the morphism of schemes corresponding to the morphism of local rings $m_\lambda \circ f$. Similarly, define a map

$$\alpha \colon \frac{k[\epsilon_1]}{\langle \epsilon_1^2 \rangle} \bigotimes_k \frac{k[\epsilon_2]}{\langle \epsilon_2^2 \rangle} \longrightarrow \frac{k[\epsilon]}{\langle \epsilon^2 \rangle},$$

by sending both $\epsilon_1$and ϵ_2 to ϵ and extend by linearity to get a morphism of local rings. Composing with the natural map

$$(f, g) \colon \mathcal{O}_{X,x} \longrightarrow \frac{k[\epsilon_1]}{\langle \epsilon_1^2 \rangle} \bigotimes_k \frac{k[\epsilon_2]}{\langle \epsilon_2^2 \rangle},$$

we get a morphism of local rings and this defines a morphism

$$\phi + \psi \colon z \longrightarrow X.$$

This defines an operation of scalar multiplication and addition of vectors, which clearly satisfy the axioms for a vector space.
(ii) If $\phi \in T_x X$ and

$$f\colon \mathcal{O}_{X,x} \longrightarrow \frac{k[\epsilon]}{\langle \epsilon^2 \rangle},$$

is the corresponding morphism of local rings, then the kernel of f contains $\mathfrak{m}^2$. On the other hand, the inverse image of $\langle \epsilon \rangle$ is by definition contained in $\mathfrak{m}$. It follows that we get a linear map of vector spaces

$$\frac{\mathfrak{m}}{\mathfrak{m}^2} \longrightarrow k\langle \epsilon \rangle \simeq k,$$

that is, an element of the dual space

$$\left(\frac{\mathfrak{m}}{\mathfrak{m}^2}\right)^*,$$

and it is not hard to see that this assignment induces a bijection.
2. 1. By assumption there are open subsets U and V and isomorphisms $\mathcal{L}|_U \simeq \mathcal{O}_U$, $\mathcal{M}|_V \simeq \mathcal{O}_V$. Passing to the open subset $U \cap V$ we may as well assume that $\mathcal{L} = \mathcal{M} = \mathcal{O}_X$. It suffices to check that the map is an isomorphism on stalks. Suppose that $x \in X$ and let $A = \mathcal{O}_{X,x}$. Then A is a local ring and we are given a surjective A-module homomorphism $\phi\colon A \longrightarrow A$. ϕ is given by multiplication by an element a of A. Suppose that $\phi(b) = 1$. Then $ab = 1$ and so a is a unit and ϕ is an isomorphism. Thus f is an isomorphism on stalks and f is an isomorphism.
2. Suppose that $m > n$. As $\dim V \leq n+1$ it follows that t_i is a linear combination of the other sections, for some $1 \leq i \leq m$. Let $\pi\colon \mathbb{P}^m \longrightarrow \mathbb{P}^{m-1}$ be the projection map which drop the ith coordinate. The composition

$$\pi \circ \phi\colon X \longrightarrow \mathbb{P}^{m-1},$$

is the morphism given by $t_0, t_1, \dots, \hat{t_i}, \dots, t_n$. So we may assume $m = n$ by induction on $m - n$.
Suppose first that $\dim |V| = \dim V - 1 = n$. In this case both $s_1, s_2, \dots, s_n$ and $t_1, t_2, \dots, t_n$ are bases of V. So there is a unique matrix $A = (a_{ij})$ such that

$$t_i = \sum a_{ij} s_j.$$

This matrix corresponds to an isomorphism $\sigma\colon \mathbb{P}^n \longrightarrow \mathbb{P}^n$ and it is clear that $\psi = \sigma \circ \phi$.
In general the image of X is contained in linear spaces Λ_i, $i = 1$ and 2 of dimension $\dim |V| = \dim V - 1$. Pick complimentary linear subspaces Λ'_i. We have already exhibited an isomorphism $\sigma_1\colon \Lambda_1 \longrightarrow \Lambda_2$, such that $\psi = \sigma_1 \circ \phi$ and we may extend this to an isomorphism of $\sigma\colon \mathbb{P}^n \longrightarrow \mathbb{P}^n$ such that $\sigma(\Lambda'_1) = \Lambda'_2$ and $\psi = \sigma \circ \phi$.

3. (a) Let $\mathcal{L} = \phi^*\mathcal{O}_{\mathbb{P}^n}(1)$. As $\operatorname{Pic}(\mathbb{P}^n) = \mathbb{Z}$ it follows that $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$, for some integer d. As $\mathcal{L}$ is globally generated $d \geq 0$. If $d = 0$ then $\phi(\mathbb{P}^n)$ is a point. Otherwise $d > 0$ and $\mathcal{L}$ is ample. Suppose that $C \subset \mathbb{P}^n$ is an irreducible curve. As $\mathcal{L}$ is ample, $\mathcal{L}|_C$ is not the trivial invertible sheaf. If $x \in C$ then we may find a section $\sigma \in H^0(\mathbb{P}^n, \mathcal{L})$ which does not vanish at x. As $\mathcal{L}|_C$ is not the trivial invertible sheaf, $\sigma|_C$ must vanish somewhere. Therefore the image of C is curve. Let $X = \phi(\mathbb{P}^n)$. If $\dim X < n$, then the fibres of $\phi\colon \mathbb{P}^n \longrightarrow X$ are positive dimensional. But then the fibres must contain curves C (just cut by hyperplanes) which are sent to a point, a contradiction.
(b) As stated, this is obviously false. Let $\phi\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^2$ be the morphism

$$[S : T] \longrightarrow [S : S : T].$$

It is clear in this case that $d = 1$. The 1-uple embedding is the identity. But then we cannot hope to project from $\mathbb{P}^1$ down to $\mathbb{P}^2$.
So let's assume that the image of ϕ is non-degenerate, that is, not contained in a hyperplane. ϕ is given by a linear system. It follows that there is an invertible sheaf $\mathcal{L}$ and a collection of sections $s_1, s_2, \dots, s_a \subset H^0(\mathbb{P}^n, \mathcal{L})$. Since $\operatorname{Pic}(\mathbb{P}^n) \simeq \mathbb{Z}$, generated by $\mathcal{O}_{\mathbb{P}^n}(1)$, it follows that $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d)$, up to isomorphism. Let $t_0, t_1, \dots, t_N$ be the standard basis of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ given by monomials of degree d. Then the induced morphism is the d-uple embedding $\mathbb{P}^n \longrightarrow \mathbb{P}^N$. Let

$$V \subset H^0(\mathbb{P}^n, \mathcal{L}),$$

be the subvector space spanned by $s_1, s_2, \dots, s_a$. Our assumption that ϕ is non-degenerate means that $s_1, s_2, \dots, s_a$ are a basis of V. We may extend this to a basis of $H^0(\mathbb{P}^n, \mathcal{L})$ and this defines an automorphism σ of $\mathbb{P}^N$. Projecting down to the first $a+1$ coordinates gives the morphism ϕ. Finally note that applying an automorphism of $\mathbb{P}^N$ is the same as projecting from the linear space L, which is the image under σ of the space spanned by the last $N - a - 1$ coordinates and an automorphism of $\mathbb{P}^n$.
4. (a) If $\mathcal{L}$ is ample then $\mathcal{L}^m$ is very ample, for some positive integer m. But then there is an immersion $X \longrightarrow \mathbb{P}^n_k$ for some positive integer n and it follows that X is separated.
(b) By assumption there are two open subsets U_1 and U_2 both of which are isomorphic to $\mathbb{A}^1_k$. Let $\mathcal{L}$ be an invertible sheaf on X and let $\mathcal{L}_i$ be the restriction of $\mathcal{L}$ to U_i. As $\operatorname{Pic}(U_i) = 0$ it follows that $\mathcal{L}_i \simeq \mathcal{O}_{U_i}$. Suppose that $\{p_1, p_2\}$ are the double points of X so that

$$X - \{p_1, p_2\} = U_i - \{p_i\}.$$

The section 1 on U_1 corresponds to a non-vanishing section $f(x)$ on U_2. It follows that $f(x) = ax^m$, for some positive integer m and a non-zero scalar a. Multiplying through by automorphisms of U_2 which fix p_2 we can assume that $a = 1$. Let's call this invertible sheaf $\mathcal{L}_m$. If we tensor $\mathcal{L}_m$ with $\mathcal{L}_n$ we get the global section 1 on U_1 and the global section $f(x) = x^{m+n}$ on U_2. It follows that $\operatorname{Pic}(X) = \mathbb{Z}$ (the inverse of the $\mathcal{L}_m$ is the invertible sheaf $\mathcal{L}_{-m}$ which has a global section which restricts to x^m on U_1 and 1 on U_2).
Now let's consider if any of these line bundles are ample. By symmetry we may suppose that $m \geq 0$. Sections of $\mathcal{L}_m$ correspond to pairs $g(x)$ on U_1 and $x^m g(x)$ on U_2, where $g(x)$ is a polynomial. There are two cases. If $m > 0$ then this section always vanishes at p_2. If $m = 0$ then this section only vanishes at p_1 if $g(x)$ has a zero at p_1, in which case the section also vanishes at p_2. Either way, $\mathcal{L}_m$ does not separate points.
5. (a) Let $\mathcal{F}$ be a coherent sheaf. By assumption there is an integer n_0 such that $\mathcal{F} \otimes \mathcal{L}^n$ is globally generated for all $n \geq n_0$. Pick $x \in X$. Then we may find $l_1, l_2, \dots, l_k \in H^0(X, \mathcal{F} \otimes \mathcal{L}^n)$ whose images generate the stalk at x. Pick $m \in \mathcal{M}$ not vanishing at x. Then $m^n l_1$, $m^n l_2$, $\dots$, $m^n l_k$ are naturally global sections of $\mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{M}^n$ which generate the stalk at x. Hence $\mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{M}^n$ is globally generated so that $\mathcal{L} \otimes \mathcal{M}$ is ample.
(b) As $\mathcal{L}$ is ample, we may pick l so that $\mathcal{M} \otimes \mathcal{L}^l$ is globally generated. If $m > 0$ is any positive integer, then

$$\mathcal{M} \otimes \mathcal{L}^{l+m} = \mathcal{M} \otimes \mathcal{L}^l \otimes \mathcal{L}^m,$$

is ample by (a). So $\mathcal{M} \otimes \mathcal{L}^n$ is ample for any $n > l$.
(c) Since $\mathcal{O}_X$ is globally generated we may find $k > 0$ so that $\mathcal{M}^k$ is globally generated. As $\mathcal{L}$ is ample then so is $\mathcal{L}^k$. But then

$$(\mathcal{L} \otimes \mathcal{M})^k = \mathcal{L}^k \otimes \mathcal{M}^k,$$

is ample by (a). It follows that

$$\mathcal{L} \otimes \mathcal{M},$$

is ample.
(d) By assumption we may find sections $l_1, l_2, \dots, l_a \in H^0(X, \mathcal{L})$ and $m_1, m_2, \dots, m_b \in H^0(X, \mathcal{M})$ such that X_{l_i} and X_{m_j} are an open affine cover of X. Consider the sections $l_i m_j \in H^0(X, \mathcal{L} \otimes \mathcal{M})$. Note that $X_{ij} = X_{l_i} \cap X_{m_j}$ is affine. Since m_j is not zero on X_{ij}, the images

$$\frac{l_{i'} m_j}{l_i m_j} = \frac{l_{i'}}{l_i},$$

generate $H^0(X_{ij}, \mathcal{O}_X)$, since the images even generate $H^0(X_{l_i}, \mathcal{O}_X)$. It follows that the sections $l_i m_j$ define an immersion of X into $\mathbb{P}^n$ into projective space such that the pullback of $\mathcal{O}_{\mathbb{P}^n}(1)$ is $\mathcal{L} \otimes \mathcal{M}$. But then $\mathcal{L} \otimes \mathcal{M}$ is very ample.
(e) First of all we know that there is a positive integer m such that $\mathcal{L}^m$ is very ample. On the other hand, by the definition of ample, we know that there is an integer m_0 such that $\mathcal{L}^n$ is globally generated for all $n \geq m_0$. Let $n_0 = m_0 + m$. If $n \geq n_0 + m$ then

$$\mathcal{L}^n = \mathcal{L}^{n-m} \otimes \mathcal{L}^m,$$

is very ample by (d).