1. (i) Call a line **standard** if it is either horizontal or vertical.

It is expedient to prove an even stronger result. We prove that if \( f: U \rightarrow \mathbb{C} \) is any function, where \( U \) is the complement of finitely many standard lines, which restricts to a polynomial on any standard line contained in \( U \), then \( f \) is a polynomial. We will be somewhat sloppy and say that a standard line is contained in \( U \) if it is not one of the deleted lines (strictly speaking, only the line minus finitely many points lies in \( U \)).

Note that if \( V \subset U \) is obtained from \( U \) by deleting finitely many more standard lines and \( f|_V \) is a polynomial, then \( f \) is a polynomial. Indeed \( f|_V \) extends to a polynomial function \( g: U \rightarrow \mathbb{C} \). If \( l \) is a line in \( U \) then \( f|_l \) and \( g|_l \) agree on an open subset of the line and so are equal. But then \( f = g \).

Let \( d \) be the smallest positive integer such that there are uncountably many real numbers \( r \) such that the restriction of \( f \) to the vertical line \( x = r \) is a polynomial of degree at most \( d \) and there are uncountably many real numbers \( s \) such that the restriction of \( f \) to the horizontal line \( y = s \) is a polynomial of degree at most \( d \).

We proceed by induction on \( d \). Suppose that \( d < 0 \), so that \( f(x,y) \) restricts to the zero function on infinitely many horizontal and infinitely many vertical lines. If \( l \) is any standard line contained in \( U \) then the restriction of \( f \) to \( l \) is a polynomial with infinitely many zeroes, so that \( f \) must be the zero function, which is represented by the zero polynomial.

Suppose that \( d \geq 0 \). Note that the change of coordinates \( x \rightarrow x - a \) does not change the property that \( U \) is the complement of finitely many standard lines, that \( f \) restricted to any standard line is a polynomial and it also does not change the value of \( d \). So we might as well assume that the \( x \)-axis is contained in \( U \) and \( f(x,0) \) is a polynomial of degree at most \( d \). Let \( g(x,y) = f(x,y) - f(x,0) \). Then the restriction of \( g(x,y) \) to every vertical line is a polynomial in \( y \) which vanishes at the origin. Let \( V \subset U \) be the set obtained by deleting the line \( y = 0 \). Let

\[
    h: V \rightarrow \mathbb{C},
\]

be the function \( h(x,y) = f(x,y)/y \). Then \( V \) is obtained from \( \mathbb{C}^2 \) by deleting finitely many standard lines, \( h(x,y) \) is a function which when restricted to any standard line in \( V \) is a polynomial, which has degree
at most $d-1$ on uncountably many standard lines. By induction $h(x, y)$ is a polynomial function. It follows that $f(x, y) = y h(x, y) + f(x, 0)$ is a polynomial function on $V$, whence on $U$. Thus $P(\mathbb{C})$ is true.

(ii) Enumerate, $c_1, c_2, \ldots$ the points of $\mathbb{Q}$ and let $h_n(y)$ (respectively $v_n(x)$) be the polynomial which vanishes on the first $n$ horizontal (respectively vertical) lines. Let

$$f(x, y) = \sum_{i=0}^{\infty} h_i(y)v_i(x).$$

It is clear that $f(x, y)$ is not a polynomial. But suppose we pick a horizontal line, given by, $y = b$. Then $b = c_n$ for some $n$ and so

$$f(x, b) = \sum_{i \leq n} h_i(b)v_i(x),$$

so that $f(x, b)$ is a polynomial. By symmetry the restriction of $f(x, y)$ to any vertical line is a polynomial. So $P(\mathbb{Q})$ fails.

(iii) Clear, from (i) and (ii) and the Lefschetz principle.

2. (i) Let $\phi \in V$, $\psi \in V$ and let $\lambda \in k$. Then

$$\phi: z \rightarrow X,$$

is a morphism of schemes over $k$, such that the unique point of $z$ goes to $x$. But then $\phi$ corresponds to a morphism of local rings over $k$,

$$f: \mathcal{O}_{X, x} \rightarrow k[\epsilon]/(\epsilon^2).$$

Similarly suppose that $\psi$ corresponds to $g$. Note that the function

$$m_\lambda: \frac{k[\epsilon]}{(\epsilon^2)} \rightarrow \frac{k[\epsilon]}{(\epsilon^2)} \quad \text{given by} \quad a + b\epsilon \rightarrow a + \lambda b\epsilon,$$

is a morphism of local rings, which is an isomorphism if and only if $\lambda \neq 0$. Let $\lambda \phi$ be the morphism of schemes corresponding to the morphism of local rings $m_\lambda \circ f$. Similarly, define a map

$$\alpha: \frac{k[\epsilon_1]}{(\epsilon_1^2)} \otimes_k \frac{k[\epsilon_2]}{(\epsilon_2^2)} \rightarrow \frac{k[\epsilon]}{(\epsilon^2)},$$

by sending both $\epsilon_1$ and $\epsilon_2$ to $\epsilon$ and extend by linearity to get a morphism of local rings. Composing with the natural map

$$(f, g): \mathcal{O}_{X, x} \rightarrow \frac{k[\epsilon_1]}{(\epsilon_1^2)} \otimes_k \frac{k[\epsilon_2]}{(\epsilon_2^2)},$$

we get a morphism of local rings and this defines a morphism

$$\phi + \psi: z \rightarrow X.$$
This defines an operation of scalar multiplication and addition of vectors, which clearly satisfy the axioms for a vector space.

(ii) If $\phi \in T_x X$ and
\[
\phi: O_{X,x} \longrightarrow k[\epsilon]/(\epsilon^2),
\]
is the corresponding morphism of local rings, then the kernel of $\phi$ contains $m^2$. On the other hand, the inverse image of $\langle \epsilon \rangle$ is by definition contained in $m$. It follows that we get a linear map of vector spaces
\[
\frac{m}{m^2} \longrightarrow k[\epsilon] \simeq k,
\]
that is, an element of the dual space
\[
\left( \frac{m}{m^2} \right)^*,
\]
and it is not hard to see that this assignment induces a bijection.

2. 1. By assumption there are open subsets $U$ and $V$ and isomorphisms $L|_U \simeq O_U$, $M|_V \simeq O_V$. Passing to the open subset $U \cap V$ we may as well assume that $L = M = O_X$. It suffices to check that the map is an isomorphism on stalks. Suppose that $x \in X$ and let $A = O_{X,x}$. Then $A$ is a local ring and we are given a surjective $A$-module homomorphism $\phi: A \longrightarrow A$. $\phi$ is given by multiplication by an element $a$ of $A$. Suppose that $\phi(b) = 1$. Then $ab = 1$ and so $a$ is a unit and $\phi$ is an isomorphism. Thus $\phi$ is an isomorphism on stalks and $f$ is an isomorphism.

2. Suppose that $m > n$. As $\dim V \leq n + 1$ it follows that $t_i$ is a linear combination of the other sections, for some $1 \leq i \leq m$. Let $\pi: \mathbb{P}^n \longrightarrow \mathbb{P}^{n-1}$ be the projection map which drop the $i$th coordinate. The composition
\[
\pi \circ \phi: X \longrightarrow \mathbb{P}^{n-1},
\]
is the morphism given by $t_0, t_1, \ldots, \hat{t}_i, \ldots, t_n$. So we may assume $m = n$ by induction on $m - n$.

Suppose first that $\dim |V| = \dim V - 1 = n$. In this case both $s_1, s_2, \ldots, s_n$ and $t_1, t_2, \ldots, t_n$ are bases of $V$. So there is a unique matrix $A = (a_{ij})$ such that
\[
t_i = \sum a_{ij} s_j.
\]
This matrix corresponds to an isomorphism $\sigma: \mathbb{P}^n \longrightarrow \mathbb{P}^n$ and it is clear that $\psi = \sigma \circ \phi$.

In general the image of $X$ is contained in linear spaces $\Lambda_i$, $i = 1$ and 2 of dimension $\dim |V| = \dim V - 1$. Pick complimentary linear subspaces $\Lambda'_i$. We have already exhibited an isomorphism $\sigma_1: \Lambda_1 \longrightarrow \Lambda_2$, such that $\psi = \sigma_1 \circ \phi$ and we may extend this to an isomorphism of $\sigma: \mathbb{P}^n \longrightarrow \mathbb{P}^n$ such that $\sigma(\Lambda'_1) = \Lambda'_2$ and $\psi = \sigma \circ \phi$. 

3. (a) Let \( \mathcal{L} = \phi^* \mathcal{O}_{\mathbb{P}^n}(1) \). As \( \text{Pic}(\mathbb{P}^n) = \mathbb{Z} \) it follows that \( \mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d) \), for some integer \( d \). As \( \mathcal{L} \) is globally generated \( d \geq 0 \). If \( d = 0 \) then \( \phi(\mathbb{P}^n) \) is a point. Otherwise \( d > 0 \) and \( \mathcal{L} \) is ample. Suppose that \( C \subset \mathbb{P}^n \) is an irreducible curve. As \( \mathcal{L} \) is ample, \( \mathcal{L}|_C \) is not the trivial invertible sheaf. If \( x \in C \) then we may find a section \( \sigma \in H^0(\mathbb{P}^n, \mathcal{L}) \) which does not vanish at \( x \). As \( \mathcal{L}|_C \) is not the trivial invertible sheaf, \( \sigma|_C \) must vanish somewhere. Therefore the image of \( C \) is curve. Let \( X = \phi(\mathbb{P}^n) \). If \( \dim X < n \), then the fibres of \( \phi: \mathbb{P}^n \to X \) are positive dimensional. But then the fibres must contain curves \( C \) (just cut by hyperplanes) which are sent to a point, a contradiction.

(b) As stated, this is obviously false. Let \( \phi: \mathbb{P}^1 \to \mathbb{P}^2 \) be the morphism

\[
[S : T] \mapsto [S : S : T].
\]

It is clear in this case that \( d = 1 \). The 1-uple embedding is the identity. But then we cannot hope to project from \( \mathbb{P}^1 \) down to \( \mathbb{P}^2 \).

So let’s assume that the image of \( \phi \) is non-degenerate, that is, not contained in a hyperplane. \( \phi \) is given by a linear system. It follows that there is an invertible sheaf \( \mathcal{L} \) and a collection of sections \( s_1, s_2, \ldots, s_a \subset H^0(\mathbb{P}^n, \mathcal{L}) \). Since \( \text{Pic}(\mathbb{P}^n) \approx \mathbb{Z} \), generated by \( \mathcal{O}_{\mathbb{P}^n}(1) \), it follows that \( \mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(d) \), up to isomorphism. Let \( t_0, t_1, \ldots, t_N \) be the standard basis of \( H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \) given by monomials of degree \( d \). Then the induced morphism is the \( d \)-uple embedding \( \mathbb{P}^n \to \mathbb{P}^N \). Let

\[ V \subset H^0(\mathbb{P}^n, \mathcal{L}), \]

be the subvector space spanned by \( s_1, s_2, \ldots, s_a \). Our assumption that \( \phi \) is non-degenerate means that \( s_1, s_2, \ldots, s_a \) are a basis of \( V \). We may extend this to a basis of \( H^0(\mathbb{P}^n, \mathcal{L}) \) and this defines an automorphism \( \sigma \) of \( \mathbb{P}^N \). Projecting down to the first \( a+1 \) coordinates gives the morphism \( \phi \). Finally note that applying an automorphism of \( \mathbb{P}^N \) is the same as projecting from the linear space \( L \), which is the image under \( \sigma \) of the space spanned by the last \( N-a-1 \) coordinates and an automorphism of \( \mathbb{P}^n \).

4. (a) If \( \mathcal{L} \) is ample then \( \mathcal{L}^m \) is very ample, for some positive integer \( m \). But then there is an immersion \( X \to \mathbb{P}^n \) for some positive integer \( n \) and it follows that \( X \) is separated.

(b) By assumption there are two open subsets \( U_1 \) and \( U_2 \) both of which are isomorphic to \( \mathbb{A}_k^1 \). Let \( \mathcal{L} \) be an invertible sheaf on \( X \) and let \( \mathcal{L}_i \) be the restriction of \( \mathcal{L} \) to \( U_i \). As \( \text{Pic}(U_i) = 0 \) it follows that \( \mathcal{L}_i \simeq \mathcal{O}_{U_i} \). Suppose that \( \{p_1, p_2\} \) are the double points of \( X \) so that

\[ X - \{p_1, p_2\} = U_i - \{p_i\}. \]
The section 1 on $U_1$ corresponds to a non-vanishing section $f(x)$ on $U_2$. It follows that $f(x) = ax^m$, for some positive integer $m$ and a non-zero scalar $a$. Multiplying through by automorphisms of $U_2$ which fix $p_2$ we can assume that $a = 1$. Let’s call this invertible sheaf $\mathcal{L}_m$. If we tensor $\mathcal{L}_m$ with $\mathcal{L}_n$ we get the global section 1 on $U_1$ and the global section $f(x) = x^{m+n}$ on $U_2$. It follows that Pic($X$) = $\mathbb{Z}$ (the inverse of the $\mathcal{L}_m$ is the invertible sheaf $\mathcal{L}_{-m}$ which has a global section which restricts to $x^m$ on $U_1$ and 1 on $U_2$).

Now let’s consider if any of these line bundles are ample. By symmetry we may suppose that $m \geq 0$. Sections of $\mathcal{L}_m$ correspond to pairs $g(x)$ on $U_1$ and $x^m g(x)$ on $U_2$, where $g(x)$ is a polynomial. There are two cases. If $m > 0$ then this section always vanishes at $p_2$. If $m = 0$ then this section only vanishes at $p_1$ if $g(x)$ has a zero at $p_1$, in which case the section also vanishes at $p_2$. Either way, $\mathcal{L}_m$ does not separate points.

5. (a) Let $F$ be a coherent sheaf. By assumption there is an integer $n_0$ such that $F \otimes \mathcal{L}^n$ is globally generated for all $n \geq n_0$. Pick $x \in X$. Then we may find $l_1, l_2, \ldots, l_k \in H^0(X, F \otimes \mathcal{L}^n)$ whose images generate the stalk at $x$. Pick $m \in \mathcal{M}$ not vanishing at $x$. Then $m^n l_1, m^n l_2, \ldots, m^n l_k$ are naturally global sections of $F \otimes \mathcal{L}^n \otimes \mathcal{M}^n$ which generate the stalk at $x$. Hence $F \otimes \mathcal{L}^n \otimes \mathcal{M}^n$ is globally generated so that $\mathcal{L} \otimes \mathcal{M}$ is ample.

(b) As $\mathcal{L}$ is ample, we may pick $l$ so that $\mathcal{M} \otimes \mathcal{L}^l$ is globally generated. If $m > 0$ is any positive integer, then

$$\mathcal{M} \otimes \mathcal{L}^{l+m} = \mathcal{M} \otimes \mathcal{L}^l \otimes \mathcal{L}^m,$$

is ample by (a). So $\mathcal{M} \otimes \mathcal{L}^n$ is ample for any $n > l$.

(c) Since $\mathcal{O}_X$ is globally generated we may find $k > 0$ so that $\mathcal{M}^k$ is globally generated. As $\mathcal{L}$ is ample then so is $\mathcal{L}^k$. But then

$$(\mathcal{L} \otimes \mathcal{M})^k = \mathcal{L}^k \otimes \mathcal{M}^k,$$

is ample by (a). It follows that

$$\mathcal{L} \otimes \mathcal{M},$$

is ample.

(d) By assumption we may find sections $l_1, l_2, \ldots, l_a \in H^0(X, \mathcal{L})$ and $m_1, m_2, \ldots, m_b \in H^0(X, \mathcal{M})$ such that $X_{l_i}$ and $X_{m_j}$ are an open affine cover of $X$. Consider the sections $l_i m_j \in H^0(X, \mathcal{L} \otimes \mathcal{M})$. Note that $X_{ij} = X_{l_i} \cap X_{m_j}$ is affine. Since $m_j$ is not zero on $X_{ij}$, the images

$$\frac{l_i m_j}{l_i} = \frac{l_i}{l_i}.$$
generate \( H^0(X_{ij}, \mathcal{O}_X) \), since the images even generate \( H^0(X_{il}, \mathcal{O}_X) \). It follows that the sections \( l_i m_j \) define an immersion of \( X \) into \( \mathbb{P}^n \) into projective space such that the pullback of \( \mathcal{O}_{\mathbb{P}^n}(1) \) is \( \mathcal{L} \otimes \mathcal{M} \). But then \( \mathcal{L} \otimes \mathcal{M} \) is very ample.

(e) First of all we know that there is a positive integer \( m \) such that \( \mathcal{L}^m \) is very ample. On the other hand, by the definition of ample, we know that there is an integer \( m_0 \) such that \( \mathcal{L}^n \) is globally generated for all \( n \geq m_0 \). Let \( n_0 = m_0 + m \). If \( n \geq n_0 + m \) then

\[
\mathcal{L}^n = \mathcal{L}^{n-m} \otimes \mathcal{L}^m,
\]

is very ample by (d).