MODEL ANSWERS TO HWK #5

8.2 Given $x \in X$, let $B_x \subset V$ be the subset of sections s such that $s_x \in \mathfrak{m}_x \mathcal{E}$. Note that there is a linear map

$$\phi_x \colon V \longrightarrow \mathcal{E}/\mathfrak{m}_x\mathcal{E},$$

which sends a section s to its class in the quotient. B_x is then the kernel of ϕ_x. As V generates $\mathcal{E}$, ϕ_x is surjective. Note that $\mathcal{E}/\mathfrak{m}_x\mathcal{E}$ is a vector space of dimension r equal to the rank of $\mathcal{E}$. Thus B_x has codimension r. Let $B \subset X \times V$ be the union of the B_x. Then B is a closed subset of $X \times V$ (where V is considered as an affine space). Let $p \colon B \longrightarrow X$ denote projection onto the first factor and $q \colon B \longrightarrow V$ denote projection onto the second factor. Then p is surjective with irreducible fibres of dimension $\dim V - r$. It follows that B has dimension $\dim V - r + n < \dim V$. $q(B)$ is a constructible subset of V. As the dimension of B is less than the dimension of V, it follows that $q(B)$ is not dense in V.
Thus we may find $s \in V$ which is not in B. But then $s_x \notin \mathfrak{m}_x\mathcal{E}$, for every $x \in X$. s gives rise to an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}' \longrightarrow 0,$$

where $\mathcal{E}'$ is defined to be the quotient. As $s_x \notin \mathfrak{m}_x\mathcal{E}$, it follows that $\mathcal{E}'$ is locally free.
8.3 (a) By virtue of (II.8.11) there is an exact sequence

$$p_1^*\Omega_{X/S} \longrightarrow \Omega_{X \underset{S}{\times} Y/S} \longrightarrow \Omega_{X \underset{S}{\times} Y/X} \longrightarrow 0.$$

By virtue of (II.8.10),

$$\Omega_{X \underset{S}{\times} Y/X} = p_2^*\Omega_{Y/S}.$$

Thus there is an exact sequence

$$p_1^*\Omega_{X/S} \longrightarrow \Omega_{X \underset{S}{\times} Y/S} \longrightarrow p_2^*\Omega_{Y/S} \longrightarrow 0.$$

By symmetry there is an exact sequence

$$p_2^*\Omega_{Y/S} \longrightarrow \Omega_{X \underset{S}{\times} Y/S} \longrightarrow p_1^*\Omega_{X/S} \longrightarrow 0.$$

Composing we get a morphism of sheaves

$$p_1^*\Omega_{X/S} = \Omega_{X \underset{S}{\times} Y/Y} \longrightarrow p_1^*\Omega_{X/S} = \Omega_{X \underset{S}{\times} Y/Y},$$

which is the identity. Thus we have a short exact sequence

$$0 \longrightarrow p_1^*\Omega_{X/S} \longrightarrow \Omega_{X\underset{S}{\times}Y/S} \longrightarrow p_2^*\Omega_{Y/S} \longrightarrow 0,$$

and so by symmetry a short exact sequence

$$0 \longrightarrow p_2^*\Omega_{Y/S} \longrightarrow \Omega_{X\underset{S}{\times}Y/S} \longrightarrow p_1^*\Omega_{X/S} \longrightarrow 0.$$

But then the injection in the first exact sequence defines a splitting of the second exact sequence.

(b) We have

$$\Omega_{X\times Y/k} \simeq p_1^*\Omega_{X/k} \oplus p_2^*\Omega_{Y/k}.$$

Now take the highest wedge product of both sides to get

$$\omega_{X\times Y} \simeq p_1^*\omega_X \otimes p_2^*\omega_Y.$$

(c) We have $\omega_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3)$. Thus

$$\omega_Y = \omega_{\mathbb{P}^2}(Y) = \mathcal{O}_Y,$$

by adjunction. It follows that $\omega_X = \mathcal{O}_X$ and so $p_g(X) = 1$. We have

$$p_a(Y) = \binom{2}{2} = 1,$$

and so

$$p_a(X) = p_a(Y)^2 - 2p_a(Y) = 1 - 2 = -1.$$

8.4 (a) Suppose that $F_1, F_2, \dots, F_r \in S$ are homogeneous polynomials which generate I. Let H_i be the hypersurface defined by F_i. Let U_j be the standard open affine $X_j \neq 0$. Then

$$f_i = \frac{F_i}{X_j^{d_i}},$$

where d_j is the degree of F_i, generates the ideal of $H_i \cap U_j$ and $f_1, f_2, \dots, f_r$ generates the ideal of $Y \cap U_j$. But then

$$\mathcal{I}_Y = \mathcal{I}_{H_1} + \mathcal{I}_{H_2} + \dots + \mathcal{I}_{H_r},$$

and so

$$Y = H_1 \cap H_2 \cap \dots \cap H_r.$$

Now suppose that

$$Y = H_1 \cap H_2 \cap \dots \cap H_r.$$

We first show that the ideal of H_i is principal; there are many ways to see this. For example, the ideal sheaf is a line bundle. Tensor by H_i to get map

$$\mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(H_i).$$

The last line bundle is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(d_i)$ for some d_i and the image of 1 is a polynomial F_i of degree d_i.

Let I be the ideal of Y and let J be the ideal generated by $F_1, F_2, \dots, F_r$, so that Y is the scheme associated to I and $J \subset I$. J has height r (since Y has codimension r) and it is generated by r elements. It follows that J is unmixed, the height of every associated prime is also r. But then these primes correspond to irreducible components of Y and so $I = J$.
(b) Let X be the cone over Y. Then X is a complete intersection in $\mathbb{A}^{n+1}$. On the other hand, Y is regular in codimension one, as it is normal, and so X is regular in codimension one as well (the singular locus of X is the cone over the singular locus of Y). But then X is normal, so that Y is projectively normal.
(c) Surjectivity follows from (II.5.14.d). If X is a projective variety note that the dimension $h^0(X, \mathcal{O}_X)$ of the k-vector space $H^0(X, \mathcal{O}_X)$ is equal to the number of connected components of X. As $H^0(Y, \mathcal{O}_Y)$ is a quotient of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = k$, it follows that Y is connected.
(d) By (c) Y is always connected. By Bertini's theorem and induction on r, if we choose $H_1, H_2, \dots, H_r$ belonging to an open subset of $|H_1| \times |H_2| \times \cdots \times |H_r|$ then Y is regular. As Y is regular and Y is connected it is irreducible. But then Y is a variety so that it is smooth.
(e) Let Z be the intersection of the first $r-1$ hypersurfaces. By induction on r,

$$K_Z = (\sum_{i=1}^{r-1} d_i - n - 1)H|_Z.$$

By adjunction, we have

$$K_Y = (K_Z + Y)|_Y = (K_Z + d_r H|_Z)|_Y = (\sum_{i=1}^{r} d_i - n - 1)H|_Y.$$

(f) In this case

$$p_g(Y) = h^0(Y, \mathcal{O}_Y(d-n-1)) = h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-n-1)) = \binom{d-1}{n}.$$

Here we use lower case to denote the dimension of the corresponding cohomology group and we use the fact that the surjection

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-n-1)) \longrightarrow H^0(Y, \mathcal{O}_Y(d-n-1)),$$

is in fact an isomorphism, as no polynomial of degree $d-n-1$ vanishes on Y.
(g) Let $C = Y$ be the intersection of two smooth surfaces of degree d and e. Let S be the first surface. Then there is an exact sequence

$$0 \longrightarrow H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(e-4)) \longrightarrow H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d+e-4)) \longrightarrow H^0(S, \mathcal{O}_S(d+e-4)) \longrightarrow 0.$$

The first term is the space of polynomials of degree $d+e-4$ vanishing on S; since any such is divisible by a polynomial of degree d, this is the same as the space of polynomials of degree $e-4$ on $\mathbb{P}^3$. Thus

$$\begin{aligned} h^0(S, \mathcal{O}_S(d+e-4)) &= h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d+e-4)) - h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(e-4)) \\ &= \binom{d+e-1}{3} - \binom{e-1}{3}. \end{aligned}$$

Note that

$$\binom{e-1}{3} = \frac{(e-1)(e-2)(e-3)}{3},$$

does indeed vanish when $e = 1$, 2 or 3. There is an exact sequence

$$0 \longrightarrow H^0(S, \mathcal{O}_S(d-4)) \longrightarrow H^0(S, \mathcal{O}_{\mathbb{P}^3}(d+e-4)) \longrightarrow H^0(C, \mathcal{O}_C(d+e-4)) \longrightarrow 0.$$

The first term is the space of polynomials of degree $d+e-4$ vanishing on C; since any such is divisible by a polynomial of degree e, this is the same as the space of polynomials of degree $d-4$ on S. Every such is the restriction of a polynomial of degree $d-4$ from $\mathbb{P}^3$ and no such polynomial can vanish on S. Putting all of this together we have

$$\begin{aligned} p_g(C) &= h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d+e-4)) - h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-4)) - h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(e-4)) \\ &= \binom{d+e-1}{3} - \binom{d-1}{3} - \binom{e-1}{3} \\ &= \frac{1}{2}de(d+e-1) + 1. \end{aligned}$$

8.5 (a) Let $U = X - Y$ and $\tilde{U} = \tilde{X} - Y'$. Then U and $\tilde{U}$ are isomorphic. As Y has codimension at least two, it follows that

$$\mathrm{Cl}(\tilde{U}) = \mathrm{Cl}(U) = \mathrm{Cl}(X).$$

On the other hand, as Y' is a prime divisor, there is an exact sequence

$$\mathbb{Z} \longrightarrow \mathrm{Cl}(\tilde{X}) \longrightarrow \mathrm{Cl}(\tilde{U}) \longrightarrow 0.$$

As X and $\tilde{X}$ are smooth,

$$\mathrm{Cl}(X) \simeq \mathrm{Pic}(X) \qquad \text{and} \qquad \mathrm{Cl}(\tilde{X}) \simeq \mathrm{Pic}(\tilde{X}),$$

so that there is an exact sequence

$$\mathbb{Z} \longrightarrow \mathrm{Pic}(\tilde{X}) \longrightarrow \mathrm{Pic}(X) \longrightarrow 0.$$

It is clear that the first map sends 1 to $\mathcal{O}_{X'}(Y')$, the normal bundle of Y' in X'. The restriction of this to a fibre gives $\mathcal{O}(-1)$. Thus we have a short exact sequence,

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Pic}(\tilde{X}) \longrightarrow \mathrm{Pic}(X) \longrightarrow 0.$$

The map

$$\pi^*\colon \operatorname{Pic}(X) \longrightarrow \operatorname{Pic}(\tilde{X}),$$

which sends an invertible sheaf to its pullback defines a splitting of this exact sequence. Thus

$$\operatorname{Pic}(X') \simeq \operatorname{Pic}(X) \oplus \mathbb{Z}.$$

(b) We know that $K_{\tilde{X}} = f^*D + qY'$ for some integer q, for some Cartier divisor on X, by (a). Restricting to $\tilde{U}$ which is isomorphic to U, we see that $D = K_X$, so that $K_{\tilde{X}} = f^*K_X + qY'$. Now let's apply adjunction on Y', to get

$$K_{Y'} = (K_{X'} + Y')|_{Y'} = (f^*K_X + (q+1)Y')|_{Y'}.$$

Let Z be the fibre of $Y' \longrightarrow Y$ over a closed point $y \in Y$. The normal bundle of Z in Y' is the pullback of the normal bundle of y in Y, so that $N_{Z/Y'}$ is locally free, of rank the dimension of Y. Thus by adjunction,

$$K_Z = K_{Y'}|_Z = (f^*K_X + (q+1)Y')|_Z.$$

Now $(f^*K_X)|_Z = 0$ (it is the pullback of a divisor from a point) and Y' restricts to $-H$, the class of a hyperplane, so that $K_Z = -(q+1)H$. On the other hand, $Z = \mathbb{P}^{r-1}$ is a toric variety, so that $K_Z = -rH$. Comparing, we get $q = r - 1$.

2. $X = X(F)$ is given by some fan F. A proper birational toric morphism $Y \longrightarrow X$ is given by repeatedly adding one dimensional rays to F and subdividing appropriately to get a fan G, so that $Y = X(G)$ is the toric variety associated to G.

Recall that if σ is a cone, then the corresponding affine toric variety U_σ is smooth if and only if the primitive vectors $v_1, v_2, \dots, v_k$ spanning the one dimensional faces of σ can be extended to a basis of the lattice N.

The first step is to reduce to the case when every cone is simplicial, that is, the vectors $v_1, v_2, \dots, v_k$ are at least independent in the vector space $N_{\mathbb{R}}$. As the faces of a simplicial cone are simplicial, it suffices to reduce to the case when every maximal (with respect to inclusion) cone is simplicial. We proceed by induction on the number d of maximal cones which are not simplicial. Suppose that σ is a maximal cone which is not simplicial. Pick a vector $v \in N$ which belongs to the interior of σ. Let F' be the fan obtained from F by inserting the ray spanned by v, and subdividing accordingly. This has the result of subdividing σ into $\sigma_1, \sigma_2, \dots, \sigma_l$ simplicial subcones, and otherwise leaves every other maximal cone unchanged. It follows that F' contains one less maximal cone which is not simplicial. After d steps, we reduce to the case when every cone in F is simplicial.

Given a simplicial cone σ, let $v_1, v_2, \dots, v_k$ be the primitive generators of its one dimensional faces. Let $V \subset N_{\mathbb{R}}$ be the vector space spanned by σ (equivalently, spanned by $v_1, v_2, \dots, v_k$), and let

$$\Lambda = \mathbb{Z}v_1 + \mathbb{Z}v_2 + \cdots + \mathbb{Z}v_k,$$

be the lattice spanned by $v_1, v_2, \dots, v_k$. Then the quotient

$$\frac{N}{\Lambda},$$

is a finitely generated abelian group. Let

$$r = r_\sigma,$$

be the cardinality of the torsion part. As noted above U_σ is smooth if and only if $r_\sigma = 1$. Let

$$r = \max_{\sigma \in F} r_\sigma,$$

be the maximum over all cones in F. We proceed by induction on r. Pick a cone τ such that $r_\tau = r$, minimal (again with respect to inclusion) with this property. Let $v_1, v_2, \dots, v_l$ be the primitive generators of the one dimensional faces of τ. Then we may find a vector w, in the interior of τ and belonging to the lattice N, whose image in N/Λ', where Λ' is the lattice spanned by $v_1, v_2, \dots, v_l$, is torsion. Consider the fan F' obtained by inserting the vector w. Let σ' be a cone in F' which is not in F. Then $\sigma' \subset \sigma \in F$, where σ' and σ have the same dimension and $\sigma \subset \tau$. If $v_1, v_2, \dots, v_k$ are the primitive generators of the one dimensional faces of σ, then, possibly relabelling, σ' has primitive generators $w, v_2, v_3, \dots, v_k$. Let Λ'' be the lattice spanned by these vectors. As the image of w in N/Λ' is non-zero and torsion, it follows that the order of the torsion part of N/Λ'' is smaller than r.
It follows by induction on the r and the number of cones τ such that $r_\tau = r$, that if we repeatedly insert vectors of the form w, then we eventually reduced to the case $r = 1$, in which case we have constructed a smooth toric variety Y, together with a toric birational morphism $Y \longrightarrow X$.