MODEL ANSWERS TO HWK #6

8.8. We follows the proof of (II.8.19). Note that if a section of a locally free sheaf vanishes on a dense open subset then the section is zero to start with.

Therefore it suffices to show that if $V \subset X$ is an open subset of X whose complement has codimension at least two then the natural restriction maps

$$H^0(X, \mathcal{O}_X(qK_X)) \longrightarrow H^0(V, \mathcal{O}_V(qK_V)) \quad \text{and} \quad H^0(X, \Omega^q_{X/k}) \longrightarrow H^0(V, \Omega^q_{V/k}),$$

are isomorphisms. For the first map, we argue exactly as in the proof of (II.8.19). For the second map, from the fact that

$$H^0(U, \mathcal{O}_U) \longrightarrow H^0(U \cap V, \mathcal{O}_{U\cap V}),$$

is a bijection, it follows that the direct sum of these maps is also bijection. As $\Omega^q_{X/k}$ is a locally free sheaf, the second map is an isomorphism as well.

2.1 (a) There is an exact sequence

$$0 \longrightarrow \mathbb{Z}_U \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_P \oplus \mathbb{Z}_Q \longrightarrow 0.$$

From the long exact sequence of cohomology we get

$$0 \longrightarrow H^0(X, \mathbb{Z}) \longrightarrow H^0(X, \mathbb{Z}_P \oplus \mathbb{Z}_Q) \longrightarrow H^1(X, \mathbb{Z}_U) \longrightarrow 0.$$

Here we used that fact $H^0(X, \mathbb{Z}_U) = 0$ and $H^1(X, \mathbb{Z}) = 0$. In this case,

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H^1(X, \mathbb{Z}_U) \longrightarrow 0,$$

so that $H^1(X, \mathbb{Z}_U) \neq 0$.

(b) By induction on n. The case $n = 1$ is (a). Let V be the complement of the first n hyperplanes and let Z be the last hyperplane. Then we have an exact sequence

$$0 \longrightarrow \mathbb{Z}_U \longrightarrow \mathbb{Z}_V \longrightarrow (\mathbb{Z}_V)_Z \longrightarrow 0.$$

The last map is the natural restriction map and one checks that $\mathbb{Z}_U$ is the kernel. Now $(\mathbb{Z}_V)|_Z$ is $\mathbb{Z}_W$, where W is the complement of n general hyperplanes in $Y = \mathbb{A}^{n-1}$. So we get an exact sequence

$$H^{n-1}(X, \mathbb{Z}_V) \longrightarrow H^{n-1}(Z, \mathbb{Z}_W) \longrightarrow H^n(X, \mathbb{Z}_U) \longrightarrow 0.$$

By induction $H^{n-1}(Y, \mathbb{Z}_W) \neq 0$ and so it is enough to show that $H^{n-1}(X, \mathbb{Z}_V) = 0$.

By way of induction, we will prove that if V is the complement of $r \leq n$ hyperplanes in $\mathbb{A}^n$ then $H^i(X, \mathbb{Z}_V) = 0$ for $i \geq \min(r - 1, 1)$. Arguing as above we reduce to the case when $r = 1$. In this case V is the complement of a single hyperplane H and there is an exact sequence

$$0 \longrightarrow \mathbb{Z}_V \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_H \longrightarrow 0.$$

As $\mathbb{Z}$ is flasque we have $H^1(X, \mathbb{Z}) = 0$ and the map on global sections is exact, and so $H^1(X, \mathbb{Z}_V) = 0$.
2.2 We have an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{K} \longrightarrow \mathcal{K}/\mathcal{O}_X \longrightarrow 0.$$

by (II.1.21.d). $\mathcal{K}$ is a locally constant sheaf on an irreducible topological space so that it is flasque. $\mathcal{K}/\mathcal{O}_X$ is a direct sum of skyscraper sheaves. A skyscraper sheaf is flasque and a direct sum of flasque sheaves is flasque. By (II.1.21.e) taking global section is exact and $H^1(X, \mathcal{K}) = 0$ as $\mathcal{K}$ is flasque, so that $H^1(X, \mathcal{O}_X) = 0$. On the other hand, X has dimension one so $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$ (or indeed, use the fact that higher cohomology of flasque is zero).
2.3 (a) Suppose we are given a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0.$$

We check that

$$0 \longrightarrow \Gamma_Y(X, \mathcal{F}) \longrightarrow \Gamma_Y(X, \mathcal{G}) \longrightarrow \Gamma_Y(X, \mathcal{H}),$$

is exact. Injectivity is clear and that the composition is zero, since $\Gamma_Y(X, \mathcal{F}) \subset \Gamma(X, \mathcal{F})$. Suppose that $g \in \Gamma_Y(X, \mathcal{G})$ is sent to zero. Then we may find $f \in \Gamma(X, \mathcal{F})$ such that g is the image of f. But it is clear that the support of f belongs to Y, so that $f \in \Gamma_Y(X, \mathcal{F})$.
(b) We check that

$$0 \longrightarrow \Gamma_Y(X, \mathcal{F}) \longrightarrow \Gamma_Y(X, \mathcal{G}) \longrightarrow \Gamma_Y(X, \mathcal{H}),$$

is exact is $\mathcal{H}$ is flasque. By (a) we only need to check the last map is surjective. But as $\mathcal{F}$ is flasque and $h \in \Gamma_Y(X, \mathcal{H})$, we may find $g \in \Gamma(X, \mathcal{G})$ mapping to h. Let g' be the restriction of g to $X - Y$. Then g' is sent to zero, since the restriction of h to U is zero. So we may find $f' \in \Gamma(U, \mathcal{F})$ mapping to g'. As $\mathcal{F}$ is flasque, we may lift f' to $f'' \in \Gamma(X, \mathcal{F})$. Let $g'' \in \Gamma(X, \mathcal{G})$ be the image of f'' and let $g_1 = g - g''$. Then g_1 maps to h and the support of g_1 is contained in Y, so that $g_1 \in \Gamma_Y(X, \mathcal{F})$.
(c) Suppose that we embed $\mathcal{F}$ into an injective sheaf $\mathcal{I}$ and let $\mathcal{G}$ be the quotient,

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{G} \longrightarrow 0.$$

If we take the long exact sequence associated to the right derived functors of Γ_Y, we use (b) and we use the fact that $H^i_Y(X, \mathcal{I}) = 0$ for $i > 0$

we get that

$$H^1_Y(X,\mathcal{F})=0 \qquad \text{and} \qquad H^{i+1}_Y(X,\mathcal{F})\simeq H^i_Y(X,\mathcal{G}).$$

But $\mathcal{I}$ is flasque so that $\mathcal{G}$ is flasque. It follows that $H^i_Y(X,\mathcal{F})=0$ for all $i>0$ by induction on i.
(d) We have an exact sequence

$$0\longrightarrow K\longrightarrow \Gamma(X,\mathcal{F})\longrightarrow \Gamma_Y(X-Y,\mathcal{F})\longrightarrow 0,$$

where K is the kernel, where the last map is surjective, as $\mathcal{F}$ is flasque. Almost by definition, $K=\Gamma_Y(X,\mathcal{F})$.
(e) Let $\mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$. Let $\mathcal{J}^i=\mathcal{I}^i|_U$. Then $\mathcal{J}^\bullet$ is an injective resolution of $\mathcal{F}|_U$. As injective sheaves are flasque we have exact sequences

$$0\longrightarrow \Gamma_Y(X,\mathcal{I}^i)\longrightarrow \Gamma(X,\mathcal{I}^i)\longrightarrow \Gamma_Y(U,\mathcal{I}|_U)\longrightarrow 0,$$

by part (d). The long exact sequence follows from the snake lemma.
(f) There are natural restriction maps,

$$\Gamma_Y(X,\mathcal{F})\longrightarrow \Gamma_Y(V,\mathcal{F}|_V),$$

which are clearly isomorphisms. As the category of abelian sheaves has enough injectives, $\Gamma_Y(V,\cdot)$ is a universal δ-functor. The isomorphisms follow by (1.4).
2.4 Let $\mathcal{I}^\bullet$ an injective resolution of $\mathcal{F}$ as constructed in (2.2). Let $Y=Y_1\cup Y_2$. Then we have short exact sequences

$$0\longrightarrow \Gamma_{Y_{12}}(X,\mathcal{F})\longrightarrow \Gamma_{Y_1}(X,\mathcal{F})\oplus\Gamma_{Y_2}(X,\mathcal{F})\longrightarrow \Gamma_Y(X,\mathcal{F})\longrightarrow 0.$$

This follows from the exact sequence on stalks and the fact that taking direct sums is exact.
3.1 A closed subscheme of an affine scheme is affine, so that if X is affine then then X_{red} is affine.
So suppose that X_{red} is affine. Let $\mathcal{F}$ be a quasi-coherent sheaf on X and let $\mathcal{N}$ be the sheaf of nilpotent elements. As X is noetherian we $\mathcal{N}^{r+1}=0$ for some positive integer r. We have a filtration

$$\mathcal{N}^r\cdot\mathcal{F}\subset\mathcal{N}^{r-1}\cdot\mathcal{F}\subset\cdots\subset\mathcal{N}\cdot\mathcal{F}\subset\mathcal{F},$$

The successive quotients are annihilated by $\mathcal{N}$, so that they are supported on X_{red}. It follows that the higher cohomology of the successive quotients so that $H^i(X,\mathcal{F})=0$ for all $i>0$. But then X affine.
3.2 If X is affine then every irreducible component is affine as the irreducible components are closed subschemes.
So suppose that the irreducible components are affine. We proceed by induction on the number of irreducible components r. If $r=1$ there is nothing to prove. Otherwise let Y be an irreducible component of

X, let $U = X - Y$ and let Z be the closure of U in X. Then Z has $r-1$ irreducible components and so it is affine by induction. There is an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}|_Z \longrightarrow 0,$$

where $\mathcal{K}$ is the kernel. Note that the support of $\mathcal{K}$ is Y. It follows that

$$H^i(X, \mathcal{F}) = 0,$$

for all $i > 0$. But then X is affine.