## MODEL ANSWERS TO HWK #7

4.1 Let $\mathcal{U} = \{U_i\}$ be an open affine cover of $X$ which is locally finite. Then $\mathcal{V} = \{V_i\}$ is an open affine cover of $Y$ which is locally finite, where $V_i = f^{-1}(U_i)$. Note that, if $I$ is a finite set of indices, then there are natural isomorphisms

$$H^0(U_I, \mathcal{F}) = H^0(V_I, f_*\mathcal{F}).$$

It follows that

$$H^*(\mathcal{U}, \mathcal{F}) \simeq H^*(\mathcal{V}, f_*\mathcal{F}).$$

But, by (II.5.8) we have that $f_*\mathcal{F}$ is quasi-coherent and so

$$H^*(X, \mathcal{F}) \simeq H^*(\mathcal{U}, \mathcal{F}) \qquad \text{and} \qquad H^*(Y, \mathcal{F}) \simeq H^*(\mathcal{V}, f_*\mathcal{F}).$$

4.2 (a) Let $M$ and $L$ be the functions fields of $X$ and $Y$ (that is, the residue fields of the generic points of $X$ and $Y$). Then $M/L$ is a finite field extension. Pick a basis $m_1, m_2, \dots, m_r$ of the $L$-vector space $M$. Let $U = \operatorname{Spec} A \subset X$ be an open affine subset of $X$. Then $M$ is the field of fractions of $A$. We may find $a_1, a_2, \dots, a_r \in A$ such that $M = K(a_1, a_2, \dots, a_r)$. Define a morphism of sheaves

$$\mathcal{O}_X{}^r \longrightarrow \mathcal{K},$$

by sending $(f_1, f_2, \dots, f_r)$ to $\sum a_i f_i$. Let $\mathcal{M}$ be the image. Then $\mathcal{M}$ is a coherent sheaf as it is the image of a coherent sheaf. Almost by definition there is a morphism of sheaves

$$\mathcal{O}_Y f_* \mathcal{O}_X \longrightarrow$$

Taking the direct sum, pushing forward the map above and composing we get a morphism of $\mathcal{O}_Y$-modules,

$$\alpha \colon \mathcal{O}_Y^r \longrightarrow f_*\mathcal{M},$$

which is an isomorphism at the generic point, since then it reduces to the vector space isomorphism,

$$L^r \longrightarrow M.$$

(b) If we apply $\mathcal{H}\mathrm{om}(\cdot, \mathcal{F})$ to $\alpha$, we get a morphism of sheaves

$$\mathcal{H}\mathrm{om}(f_*\mathcal{M}, \mathcal{F}) \longrightarrow \mathcal{F}^r,$$

which is certainly an isomorphism at the generic point. Note that

$$\mathcal{H}\mathrm{om}(f_*\mathcal{M}, \mathcal{F}),$$

is a coherent $\mathcal{A} = f_*\mathcal{O}_X$-module. By (5.17e), there is a coherent $\mathcal{O}_X$-module $\mathcal{G}$ such that

$$f_*\mathcal{G} = \mathcal{H}\mathrm{om}(f_*\mathcal{M}, \mathcal{F}).$$

(c) Let $Y' \subset Y$ be a closed subscheme, let $X' \subset f^{-1}(Y')$ be a closed subset of the inverse such that the induced morphism $f' \colon X' \longrightarrow Y'$ is surjective. Note that $X' \subset X$ is affine, and $f'$ is a finite morphism. Indeed, to check $f'$ is finite, we may assume that $Y = \operatorname{Spec} B$ is affine and by assumption $A$ is a finitely generated $B$-module. If $I$ and $J$ are the defining ideals of $X'$ and $Y'$, then $X' = \operatorname{Spec} A/I$, $Y' = \operatorname{Spec} B/J$ and it is clear that $A/I$ is a finitely generated $B/J$-module.
Note that $f_{\mathrm{red}} \colon X_{\mathrm{red}} \longrightarrow Y_{\mathrm{red}}$ is a surjective finite morphism of noetherian, separated and reduced schemes. As (3.1) implies that $Y$ is affine if and only if $Y_{\mathrm{red}}$ is affine, we may assume that $X$ and $Y$ are reduced. Suppose that $Y' \subset Y$ is an irreducible component of $Y$. As $f$ is surjective, there is an irreducible component $X'$ of $X$ which surjects to $Y'$. The induced morphism $f' \colon X' \longrightarrow Y$ is a surjective finite morphism of noetherian, separated and integral schemes. As (3.2) implies that $Y$ is affine if and only if each irreducible component $Y'$ is affine, we may assume that $X$ and $Y$ are integral. Let $\mathcal{F}$ be a quasi-coherent sheaf on $Y$. We check that

$$H^i(Y, \mathcal{F}) = 0,$$

for all $i > 0$. By Noetherian induction and (3.7), we may suppose that

$$H^i(Y', \mathcal{G}) = 0,$$

for all proper closed subsets and all quasi-coherent sheaves $\mathcal{G}$.
By (b), we may find an exact sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow f_*\mathcal{G} \longrightarrow \mathcal{F}^r \longrightarrow \mathcal{Q} \longrightarrow 0,$$

where $\mathcal{R}$ and $\mathcal{Q}$ are quasi-coherent sheaves, supported on proper closed subsets of $Y$. By induction,

$$H^i(Y, \mathcal{F}^r) = H^i(Y, f_*\mathcal{G}),$$

and the last group is isomorphic to

$$H^i(X, \mathcal{G}),$$

by (4.1). But this vanishes as $X$ is affine and $\mathcal{G}$ is quasi-coherent. Thus

$$H^i(Y, \mathcal{F}) = 0,$$

for all $i > 0$ and all quasi-coherent sheaves $\mathcal{F}$, and so $Y$ is affine by (3.7).
4.3 Let $\mathcal{U} = \{U_x, U_y\}$, where $U_x$ is the complement of the $x$-axis and $U_y$ is the complement of the $y$-axis. Then $U_x$ and $U_y$ are both isomorphic

to $\mathbb{A}^1 \times (\mathbb{A}^1 - \{0\})$, so that they are both affine. The intersection of $U_x$ and $U_y$ is $(\mathbb{A}^1 - \{0\}) \times (\mathbb{A}^1 - \{0\})$, which is again affine. As $\mathcal{O}_X$ is coherent, we have an isomorphism,

$$H^1(\mathcal{U}, \mathcal{O}_X) \simeq H^1(X, \mathcal{O}_X).$$

Now an element of $C^1(\mathcal{U}, \mathcal{O}_X)$ is nothing but a section of $H^0(U_x \cap U_y, \mathcal{O}_X)$. Since there are no triple intersections, every cochain is automatically a cocycle, so that

$$Z^1(\mathcal{U}, \mathcal{O}_X) = C^1(\mathcal{U}, \mathcal{O}_X) = k[x, y]_{xy}.$$

Now

$$C^0(\mathcal{U}, \mathcal{O}_X) = H^0(U_x, \mathcal{O}_X) \oplus H^0(U_y, \mathcal{O}_X).$$

Note that

$$H^0(U_x, \mathcal{O}_X) = k[x, y]_x \qquad \text{and} \qquad H^0(U_y, \mathcal{O}_X) = k[x, y]_y.$$

Thus

$$B^1(\mathcal{U}, \mathcal{O}_X) = k[x, y]_x + k[x, y]_y.$$

It follows that a basis of

$$H^1(X, \mathcal{O}_X),$$

is given by monomials of the form $x^i y^j$, where $i < 0$ and $j < 0$. In particular,

$$h^1(X, \mathcal{O}_X),$$

is not finite.

It is also interesting to calculate $H^1(X, \mathcal{O}_X)$ using the fact that $X$ is toric. The fan $F$ corresponding to $X$ is the union of the two one dimensional cones spanned by $e_1$ and $e_2$ (but not including the cone spanned by $e_1$ and $e_2$) and the origin (which is a face of both one dimensional cones). Then the support of the fan $F$ is

$$|F| = \{\, (x, 0) \mid x \geq 0 \,\} \cup \{\, (0, y) \mid y \geq 0 \,\}.$$

The 0 divisor is $T$-Cartier and corresponds to the zero function on $F$. According to (15.6),

$$H^1(X, \mathcal{O}_X),$$

decomposes as a direct sum of eigenspaces, indexed by $u \in M$, where each piece is given by a local cohomology group,

$$H^1_{Z(u)}(|F|, \mathbb{C}).$$

The last group is isomorphic to the relative cohomology of the pair

$$H^1(|F|, Z(u), \mathbb{C}).$$

The long exact sequence for the pair $Z(u) \subset |F|$ is:

$$0 \longrightarrow H^0(|F|, |F| - Z(u), \mathbb{C}) \longrightarrow H^0(|F| - Z(u), \mathbb{C}) \longrightarrow H^0(|F|, \mathbb{C}) \dashrightarrow$$
$$\dashrightarrow H^1(|F|, |F| - Z(u), \mathbb{C}) \longrightarrow H^1(|F| - Z(u), \mathbb{C}) \longrightarrow H^1(|F|, \mathbb{C}) \longrightarrow 0.$$

Note that $H^0(|F|, \mathbb{C}) = \mathbb{C}$ and $H^1(|F| - Z(u), \mathbb{C})$ is always trivial. It follows that

$$H^1_{Z(u)}(|F|, \mathbb{C}),$$

is non-trivial, equal to $\mathbb{C}$, if and only if $|F| = Z(u)$, if and only if $u = (i, j)$, where $i \leq 0$ and $j \leq 0$.

4.5 As in the hint any invertible sheaf $\mathcal{L}$ determines an element $l_{\mathcal{U}}$ of $H^1(\mathcal{U}, \mathcal{O}^*_X)$, where $\mathcal{L}|_{U_i}$ is trivial. If $\mathcal{V}$ is a refinement of $\mathcal{U}$, then $\mathcal{L}|_{V_j}$ is certainly trivial, where $V_j \subset U_i$, and it is easy to check that

$$l_{\mathcal{V}} \in H^1(\mathcal{V}, \mathcal{O}^*_X),$$

is the same as the image of $l_{\mathcal{U}}$ under the natural map

$$H^1(\mathcal{U}, \mathcal{O}^*_X) \longrightarrow H^1(\mathcal{V}, \mathcal{O}^*_X).$$

Thus $\mathcal{L}$ determines an element of the direct limit. Using (5.4) this gives us a map

$$\pi\colon \operatorname{Pic}(X) \longrightarrow H^1(X, \mathcal{O}^*_X).$$

If $\mathcal{L}$ and $\mathcal{M}$ are two invertible sheaves, then there is a common cover $\mathcal{U}$ over which they are both trivial. It is easy to see that the image of $\mathcal{L} \underset{\mathcal{O}_X}{\otimes} \mathcal{M}$ in $H^1(\mathcal{U}, \mathcal{O}^*_X)$ is $l_{\mathcal{U}} + m_{\mathcal{U}}$. But then $\pi$ is a group homomorphism. To give an element of $H^1(X, \mathcal{O}^*_X)$ is to give an element of $H^1(\mathcal{U}, \mathcal{O}^*_X)$, for some open cover $\mathcal{U}$. Using this 1-cocycle, one can construct an invertible sheaf, $\mathcal{L}$, which represents this 1-cocycle. Thus $\pi$ is surjective. Suppose that $\mathcal{L}$ is sent to zero. Then there is some open cover $\mathcal{U}$ for which the corresponding 1-cocycle is a coboundary, represented by

$$\sigma_i \in H^0(U_i, \mathcal{O}^*_X).$$

But then $\sigma$ defines a global non-vanishing section of $\mathcal{L}$, so that

$$\mathcal{L} \simeq \mathcal{O}_X.$$

It follows that $\pi$ is injective.

4.7 The union of the standard open affine subsets $U_1$ and $U_2$ contains every point of $\mathbb{P}^2$ except $[1:0:0]$. $U_{12} = U_1 \cap U_2$ is affine so that $U$, $V$ and $U \cap V$ are affine subsets of $X$. Therefore $\mathcal{U} = \{U, V\}$ is certainly an open affine cover of $X$.

Let $x = x_0/x_2$ and $y = x_1/x_2$. Then $u = x_0/x_1 = xy^{-1}$ and $v = x_2/x_1 = y^{-1}$. We want to calculate the cohomology of the complex

$$\frac{k[u,v]}{\langle f(u,1,v)\rangle} \oplus \frac{k[x,y]}{\langle f(x,y,1)\rangle} \longrightarrow \frac{k[x,y,v]}{\langle f(x,y,1)}$$

We already know the kernel consists only of the constants. We check this explicitly.

Suppose that $(g, h)$ is sent to zero. Then

$$g(u, v) - h(x, y) = \alpha f(x, y, 1),$$

for some polynomial $\alpha$ in $x$, $y$ and $y^{-1}$. Note that the coefficient of $x^d$ is non-zero in $f$, since $[1 : 0 : 0]$ is not a point of the curve. We may write

$$\alpha = \beta + \gamma + \delta,$$

where $\beta$ is a polynomial whose only non-zero coefficients are of the form $x^i y^j$ where $i \leq -d - j$, for $\gamma$ we have $j \geq 0$ and $\delta$ is what is left over, namely $j < 0$ and $i > -d - j$.

Note that $(g - \beta f, h + \gamma f)$ represents the same element of

$$\frac{k[u,v]}{\langle f(u,1,v)\rangle} \oplus \frac{k[x,y]}{\langle f(x,y,1)\rangle},$$

as $(g, h)$. As $\beta f$ and $\gamma f$ only have the constant term in common, it is enough to show $\delta = 0$.

Consider the set

$$N = \{\, (i, j) \,|\, \text{the coefficient of } x^i y^j \text{ is non-zero in } \delta \,\}.$$

Pick $(i, j) \in N$ with $i$ maximal and $j$ minimal. Then the coefficient of $x^{i+d} y^j$ is non-zero in $g(u, v) - h(x, y) - \beta f - \gamma f$. So either $i + d \leq -j$ or $j \geq 0$, both of which are impossible as $(i, j) \in N$.

Thus the kernel of the map is the space of constant polynomials.

We consider the cokernel. Each element of the cokernel corresponds to a polynomial in $k[x, y, y^{-1}]$. Monomials of the form $x^i y^j$ where $j \geq 0$ are in the image as are monomials of the form $x^i y^j$, where $i \leq -j$. Since the coefficient of $x^d$ is non-zero in $f(x, y, 1)$, it follows that elements of the cokernel are a sum of monomials $x^i y^j$ where $0 \leq i < d$ and $-i < j < 0$. There are

$$\frac{d-1}{d-2}2,$$

such monomials, and we have already checked that none of these monials are in the image of the coboundary map.