MODEL ANSWERS TO HWK #8

5.1 We can split the long exact sequence of cohomology into one short exact sequence,

$$0 \longrightarrow H^0(X, \mathcal{F}') \longrightarrow H^0(X, \mathcal{F}) \longrightarrow Q \longrightarrow 0,$$

and one long exact sequence, which starts with

$$0 \longrightarrow Q' \longrightarrow H^1(X, \mathcal{F}') \longrightarrow H^1(X, \mathcal{F}) \dots,$$

where

$$Q' = \frac{H^0(X, \mathcal{F}'')}{Q}.$$

We have

$$h^0(X, \mathcal{F}) = h^0(X, \mathcal{F}') + \dim_k Q,$$

and, by an obvious induction,

$$\sum_{i\geq 1}(-1)^{i-1}h^i(X, F) = \sum_{i\geq 1}(-1)^{i-1}h^i(X, F') + \sum_{i\geq 0}(-1)^{i-1}h^i(X, F'') - \dim_k Q.$$

Adding the two equations together gives the result.

5.2 (a) Pick a divisor Y belonging to the linear system determined by $\mathcal{O}_X(1)$. Note that there is a morphism of sheaves

$$\mathcal{F}(-1) \longrightarrow \mathcal{F},$$

which is locally given by multiplication by the defining equation of Y, so that this map is an isomorphism away from Y. We get an exact sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{F}(-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{Q} \longrightarrow 0,$$

where $\mathcal{Q}$ and $\mathcal{R}$ are defined to fix exactness. Note that $\mathcal{Q}$ and $\mathcal{R}$ are coherent and they are both supported on Y. If we tensor this exact sequence by $\mathcal{O}_X(n)$ we get

$$0 \longrightarrow \mathcal{R}(n) \longrightarrow \mathcal{F}(n-1) \longrightarrow \mathcal{F}(n) \longrightarrow \mathcal{Q}(n) \longrightarrow 0,$$

By (5.1) we have

$$\Delta\chi(\mathcal{F}(n)) = \chi(\mathcal{Q}(n)) - \chi(\mathcal{R}(n)).$$

By Noetherian induction the RHS is a polynomial and so $\chi(\mathcal{F}(n))$ is also a polynomial.

(b) By Serre vanishing, there is an integer n_0 such that

$$\chi(\mathcal{F}(n)) = h^0(\mathbb{P}^n, \mathcal{F}(n)),$$

for $n \geq n_0$. But we have already seen that the RHS is precisely the dimension of the nth graded piece of $\Gamma_*(\mathcal{F})$.
5.3 (a) If X is integral, and k is an algebraically closed field, then there is a projective variety X' such that $t(X') = X$. We have that

$$H^0(X', \mathcal{O}_{X'}) = H^0(X, \mathcal{O}_X).$$

But by (I.3.4), the LHS is isomorphic to k.
(b) Clear from (5.2).
(c) Let $f \colon C \dashrightarrow X$ be a rational map from a smooth curve to a projective variety. Then f is a morphism. Thus if $f \colon C_1 \dashrightarrow C_2$ is a birational map, then f is in fact an isomorphism. It is then clear that $p_a(C)$ is a birational invariant.
If C is a smooth plane curve of degree d then the arithmetic genus of C is

$$\binom{d-1}{2}.$$

In particular, if $d \geq 3$, the arithmetic genus of C is non-zero, so that C is not rational.
5.7 (a) Let $\mathcal{F}$ be any coherent sheaf on Y. Then $\mathcal{G} = i_*\mathcal{F}$ is a coherent sheaf on X. As $\mathcal{L}$ is ample, there is an integer n_0 such that if $n \geq n_0$, then

$$H^i(X, \mathcal{G} \otimes \mathcal{L}^n) \qquad \text{for any} \qquad n \geq n_0, i > 0.$$

On the other hand,

$$H^i(Y, \mathcal{F} \otimes i^*\mathcal{L}^n) = H^i(X, \mathcal{G} \otimes \mathcal{L}^n).$$

(b) Since X_{red} is a closed subscheme, (a) implies that $\mathcal{L}_{\text{red}}$ is ample. Now suppose that $\mathcal{L}_{\text{red}}$ is ample. Let $\mathcal{F}$ be a quasi-coherent sheaf on X and let $\mathcal{N}$ be the sheaf of nilpotent elements. Then

$$\mathcal{N}^k \cdot \mathcal{L} = 0,$$

for some $k > 0$. Let $\mathcal{G} = \mathcal{N} \cdot \mathcal{F}$. By induction on k, there is a constant n_0 such that

$$H^i(X, \mathcal{G} \otimes \mathcal{L}^n) = 0,$$

for all $n \geq n_0$. There is a short exact sequence,

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathcal{H} \longrightarrow 0,$$

where $\mathcal{H}$ is supported on X_{red}. Possibly increasing n_0, we may assume that

$$H^i(X, \mathcal{H} \otimes \mathcal{L}^n) = 0,$$

for all all $n \geq n_0$. Tensoring by $\mathcal{L}^n$ and taking the long exact sequence of cohomology, we get

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0,$$

all $n \geq n_0$. But then $\mathcal{L}$ is ample by (5.3).
(c) As X_i is a closed subscheme of X, (a) implies that $\mathcal{L} \otimes \mathcal{O}_{X_i}$ is ample. Let $\mathcal{I}$ be the ideal sheaf of X_1. Let $\mathcal{F}$ be a quasi-coherent sheaf. Then there is an exact sequence

$$0 \longrightarrow \mathcal{I} \cdot \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0,$$

where $\mathcal{G}$ is a quasi-coherent sheaf supported on X_1. Tensoring by a sufficiently high power of $\mathcal{L}$ and by induction on the number of irreducible components, taking the long exact sequence of cohomology, we get that

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0,$$

all $n \geq n_0$. But then $\mathcal{L}$ is ample by (5.3).
(d) If $\mathcal{L}$ is ample, and $\mathcal{F}$ is a quasi-coherent sheaf on X, then $f_*\mathcal{F}$ is quasi-coherent sheaf on Y and

$$H^i(X, \mathcal{F} \otimes f^*\mathcal{L}^n) = H^i(Y, f_*\mathcal{F} \otimes \mathcal{L}^n) = 0,$$

for all n sufficiently large. Hence $f^*\mathcal{L}$ is ample.
For the other direction, by (b) and (c) we may suppose that X and Y are integral. Let $\mathcal{F}$ be a quasi-coherent sheaf on Y. As in the proof of (4.2), we may find an exact sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow f_*\mathcal{G} \longrightarrow \mathcal{F}^r \longrightarrow \mathcal{Q} \longrightarrow 0,$$

where $\mathcal{R}$ and $\mathcal{Q}$ are quasi-coherent sheaves, supported on proper closed subsets of Y, and $\mathcal{G}$ is a coherent sheaf on X. Tensoring by a high power of $\mathcal{L}$, applying Noetherian induction, we get

$$H^i(Y, \mathcal{F}^r \otimes \mathcal{L}^n) = H^i(X, \mathcal{G} \otimes f^*\mathcal{L}^n) = 0,$$

for all $i > 0$. But then $\mathcal{L}$ is ample.