HWK #4, DUE WEDNESDAY 02/05

Let p_1, p_2, p_3 and p_4 be four points in $\mathbb{P}^1$.
1. Let $G \subset \mathrm{PGL}(2)$ be the subgroup fixing the set of four points. Show that G is naturally a subset of S_4, which always contains the Viergrüppe (that is, the subgroup of S_4 generated by the permutations of type $(2,2)$).
2. Show that the only invariant of the four unordered points is given by the j-invariant,

$$j = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}$$

(somewhat surprisingly the factor of 2^8 is put there by number theorists to make things come out in characteristic 2). Identify the possible values of the j-invariant, for four distinct points.
3. Identify all possible quotient groups G/V and the corresponding j-invariants.
4. Show that the set of n unordered points in $\mathbb{P}^1$ is naturally identified with $\mathbb{P}^n$ (*Hint, associate to n unordered points, the polynomial which defines them*).
5. Show that the j-invariant extends to a rational map

$$j \colon \mathbb{P}^4 \dashrightarrow \mathbb{P}^1$$

which is defined at any point where the set $\{p_1, p_2, p_3, p_4\}$ has cardinality at least three.
6. Let $p \in \mathbb{P}^3$ and let $H \subset \mathbb{P}^3$ be a plane. Show that the locus

$$\Sigma_{p,H} = \{\, [l] \in \mathbb{G}(1,3) \,|\, p \in l \subset H \,\} \subset \mathbb{G}(1,3) \subset \mathbb{P}^5,$$

is a line in $\mathbb{P}^5$. Show conversely that any line lying on $\mathbb{G}(1,3)$ under the Plücker embedding is of this form.
7. Let $p \in \mathbb{P}^3$ and let $H \subset \mathbb{P}^3$ be a plane. Show that the loci

$$\Sigma_p = \{\, [l] \in \mathbb{G}(1,3) \,|\, p \in l \,\} \subset \mathbb{G}(1,3) \subset \mathbb{P}^5,$$

and

$$\Sigma_H = \{\, [l] \in \mathbb{G}(1,3) \,|\, l \subset H \,\} \subset \mathbb{G}(1,3) \subset \mathbb{P}^5,$$

are planes in $\mathbb{P}^5$. Show conversely that any plane lying on $\mathbb{G}(1,3)$ under the Plücker embedding is of this form.