1. Examples of fibre products

It turns out that the fibre product is extremely useful.

Definition 1.1. *Let $f\colon X \longrightarrow S$ be a morphism of schemes, and let $s \in S$ be a point of S. The* ***fibre over*** *s is the fibre product over the morphism f and the inclusion of s in S, where the point s is given a scheme structure by taking the residue field $\kappa(s)$.*

It is interesting to see what happens in some specific examples. First consider a family of conics in the plane,

$$X = \operatorname{Spec} \frac{k[x, y, t]}{\langle ty - x^2 \rangle}.$$

The inclusion

$$k[t] \longrightarrow \frac{k[x, y, t]}{\langle ty - x^2 \rangle},$$

realises X as a family over the affine line over k,

$$f\colon X \longrightarrow \mathbb{A}^1_k.$$

Pick a point $p \in \mathbb{A}^1$. If the point is closed, this is the same as picking a scalar, and of course the residue field is nothing more than k. If we pick a non-zero scalar a, then we just get the conic defined by $ay - x^2$ in $k[x, y]$ (since tensoring by k won't change anything),

$$X_p = \operatorname{Spec} \frac{k[x, y]}{\langle ay - x^2 \rangle}.$$

But now suppose that $a = 0$. In this case the above reduces to

$$X_0 = \operatorname{Spec} \frac{k[x, y]}{\langle x^2 \rangle},$$

a double line. It is also interesting to consider the fibre over the generic point ξ, corresponding to the maximal ideal $\langle 0 \rangle$. In this case the residue field is $k(t)$, and the **generic fibre** is

$$X_\xi = \operatorname{Spec} \frac{k(t)[x, y]}{\langle ty - x^2 \rangle},$$

which is the conic $V(ty - x^2) \subset \mathbb{A}^2_{k(t)}$ over the field $k(t)$.

Similarly, if we pick the family

$$X = \operatorname{Spec} \frac{k[x, y, t]}{\langle xy - t \rangle}.$$

then, for $a \neq 0$, the fibre is a smooth conic, but for $t = 0$ the fibre is a pair of lines.

Once again, the point is that there are some more exotic examples, which can be treated in a similar fashion. Consider for example $\mathbb{A}^1_{\mathbb{Z}} = \operatorname{Spec}\mathbb{Z}[x]$. Once again this is a scheme over $\operatorname{Spec}\mathbb{Z}$, and once again it is interesting to compute the fibres. Suppose first that we take the generic point. Then this has residue field $\mathbb{Q}$. If we tensor $\mathbb{Z}[x]$ by $\mathbb{Q}$, then we get $\mathbb{Q}[x]$. If we take Spec of this, we get the affine line over $\mathbb{Q}$. Now suppose that we take a maximal ideal $\langle p\rangle$. In this case the residue field is $\mathbb{F}_p$ the finite field with p elements. Tensoring by this field we get $\mathbb{F}_p[x]$ and taking Spec we get the affine line over the finite field with p elements.

It is also possible to figure out all the prime ideals in $\mathbb{Z}[x]$. They are

(1) $\langle 0\rangle$
(2) $\langle p\rangle$, p a prime number.
(3) $\langle f(x)\rangle$, $f(x)$ irreducible over $\mathbb{Q}$, with content one,
(4) maximal ideals of the form $\langle p, f(x)\rangle$, where $f(x)$ is a polynomial, with content one, whose reduction modulo p is irreducible.

Note that the zero ideal is the generic point, and the closure of the ideal $\langle p\rangle$ is the fibre over the same ideal downstairs. The closure of an ideal of type (3) is perhaps the most interesting. It will consist of all maximal points $\langle p, g\rangle$, where the reduction of $g(x)$ is a factor of the reduction of $f(x)$ inside $\mathbb{F}_p[x]$.

It is now possible to consider closed subschemes of $\mathbb{A}^1_{\mathbb{Z}}$. For example consider

$$X = \operatorname{Spec}\frac{\mathbb{Z}[x]}{\langle 3x-16\rangle}.$$

Fibre by fibre, we get a collection of subschemes of $\mathbb{A}^1_{\mathbb{F}_p}$. If we reduce modulo 5, that is, tensor by $\mathbb{F}_5$ then we get

$$X = \operatorname{Spec}\frac{\mathbb{F}_5[x]}{\langle 3x-1\rangle},$$

a single point. However something strange happens over the prime 3, since we get an equation which cannot be satisfied. If we think of this as the graph of the rational map 16/3, then we have a pole at 3, which cannot be removed. Of course over 2, this rational function is zero.

Now suppose that we consider x^2-3. Then we get a conic. In fact, this is the same as considering

$$\frac{\mathbb{Z}[x]}{\langle x^2-3\rangle} = \mathbb{Z}[\sqrt{3}].$$

So the seemingly strange picture we had before becomes a little more clear.

Consider the residue fields. Recall that there are three cases.

(1) If p divides the discriminant of $K/\mathbb{Q}$ (which in this case is 12), that is $p = 2$ or 3, then the ideal $\langle p \rangle$ is a square in A.

$$\langle 2 \rangle A = (\langle 1 + \sqrt{3} \rangle)^2,$$

and

$$\langle 3 \rangle A = (\langle \sqrt{3} \rangle)^2.$$

(2) If 3 is a square modulo p, the prime $\langle p \rangle$ factors into a product of distinct primes,

$$\langle 11 \rangle A = \langle 4 + 3\sqrt{3} \rangle \langle 4 - 3\sqrt{3} \rangle,$$

or

$$\langle 13 \rangle A = \langle 4 + \sqrt{3} \rangle \langle 4 - \sqrt{3} \rangle,$$

(3) If $p > 3$ and 3 is not a square mod p (e.g $p = 5$ and 7), the ideal $\langle p \rangle$ is prime in A.

Let us consider the coordinate rings in all three cases. In the first case we get

$$A/\mathfrak{p}^2,$$

and the residue field is $\mathbb{F}_p$. In the second case there are two points with coordinate rings $\mathbb{F}_p$. Finally in the third case there is a single point with coordinate ring

$$\mathbb{F}_{p^2},$$

the unique finite field with p^2 elements. Note that in all three cases, the coordinate ring of the inverse image has length two over the coordinate ring of the base (in our case $\mathbb{F}_p$). In fact this is the general picture. Finite maps have a degree, and the length of the coordinate ring over the base is equal to this degree.

Now suppose that we consider a plane conic in $\mathbb{A}^2_{\mathbb{Z}}$,

$$X = \operatorname{Spec} \frac{\mathbb{Z}[x, y]}{\langle x^2 - y^2 - 5 \rangle}.$$

Over the typical prime, we get a smooth conic in the corresponding affine plane over a finite field. But now consider what happens over $\langle 2 \rangle$ and $\langle 5 \rangle$. Modulo two, we have

$$x^2 - y^2 - 5 = (x + y + 1)^2,$$

and modulo 5 we have

$$x^2 - y^2 - 5 = (x - y)(x + y).$$

Thus we get a double line over $\langle 2 \rangle$ and a pair of lines over $\langle 5 \rangle$.

Another useful way to think of the fibre product, is as a base change. In arithmetic, one always wants to compare what happens over different fields, or even different rings.

Definition 1.2. *Let S be a scheme. As $\operatorname{Spec}\mathbb{Z}$ is a terminal object in the category of schemes, there is a unique morphism $S \longrightarrow \operatorname{Spec}\mathbb{Z}$* ***Affine*** n***-space*** *over S is the scheme obtained by base change from $\mathbb{A}^n_{\mathbb{Z}} = \operatorname{Spec}\mathbb{Z}[x_1, x_2, \dots, x_n]$, so that*

$$\mathbb{A}^n_S = \mathbb{A}^n_{\mathbb{Z}} \underset{\operatorname{Spec}\mathbb{Z}}{\times} S.$$

Now consider an interesting example over a non-algebraically close field. Consider the inclusion $\mathbb{R} \longrightarrow \mathbb{C}$. This gives a morphism of schemes,

$$f\colon X = \operatorname{Spec}\mathbb{C} \longrightarrow Y = \operatorname{Spec}\mathbb{R},$$

where X and Y are schemes with only one point, but the first has sheaf of rings given by $\mathbb{C}$ and the second $\mathbb{R}$. Now consider what happens when we make the base change f over f. Then we get a scheme

$$X \underset{Y}{\times} X.$$

Note that this has degree two over X. Since $\mathbb{C}$ is algebraically closed, in fact this must consist of two points, even though f only has one point in the fibre. Algebraically,

$$\mathbb{C} \underset{\mathbb{R}}{\otimes} \mathbb{C} \simeq \mathbb{C}^2,$$

and the spectrum has two points.

In particular, the property of being irreducible is not preserved by base change. Consider also the example of $\operatorname{Spec} k[x,t]/\langle x^2 - t\rangle \subset \mathbb{A}^2_k$ over the affine line, with coordinate t, say over an algebraically closed field k. Then the fibre over every closed point, except zero, is reducible. But the fibre over the generic point is irreducible, since $x^2 - t$ won't factor, even if you invert every polynomial in t. However suppose that we make a base change of the affine line by the affine line given by

$$\mathbb{A}^1_k \longrightarrow \mathbb{A}^1_k \qquad \text{given by} \qquad t \longrightarrow t^2.$$

After base change, the new scheme is given by $x^2 - t^2$. But this factors, even over the generic point

$$x^2 - t^2 = (x - t)(x + t).$$

Definition 1.3. *Let X be a scheme over a field k. We say that X is* ***geometrically irreducible*** *if $X \underset{\operatorname{Spec} k}{\times} \operatorname{Spec}\bar{k}$ is irreducible.*

Note that the property of being geometrically irreducible is preserved under base change.