15. Cubics I

In this section we give a geometric application of some of the ideas of the previous sections. Recall the definition of a rational variety.

Definition 15.1. A variety $X$ over $\text{Spec } k$ is rational if it birational to $\mathbb{P}^n_k$, for some $n$.

Theorem 15.2. Every smooth cubic $C \subset \mathbb{P}^2$ is irrational.

We will prove (15.2) later.

Theorem 15.3. Every smooth cubic surface $S \subset \mathbb{P}^3$ is rational.

The key to the proof of (15.3) is the following celebrated:

Theorem 15.4. Every smooth cubic surface $S \subset \mathbb{P}^3$ contains twenty seven lines.

Example 15.5. Let $S \subset \mathbb{P}^3$ be the cone over a cubic curve $C \subset \mathbb{P}^2$. Then $S$ contains infinitely many lines.

Lemma 15.6. Every cubic surface $S \subset \mathbb{P}^3$ contains a line.

Proof. A cubic is specified by choosing the coefficients of a homogeneous cubic in four variables of which there are $\binom{6}{3} = 20$; the space of all cubics is therefore naturally parametrised by $\mathbb{P}^{19}$. Consider the incidence correspondence

$$\Sigma = \{ (l, F) \in \mathbb{G}(1,3) \times \mathbb{P}^{19} | l \subset V(F) \} \subset \mathbb{G}(1,3) \times \mathbb{P}^{19}.$$ 

This is a closed subset of $\mathbb{G}(1,3) \times \mathbb{P}^{19}$ and the two natural projections $f: \Sigma \rightarrow \mathbb{G}(1,3)$ and $g: \Sigma \rightarrow \mathbb{P}^{19}$ are proper, since they are projective.

Let $l \in \mathbb{G}(1,3)$ and consider $f^{-1}(l)$. This is the space of cubics containing the line $l$. There are two ways to figure out what the fibre looks like.

One can change coordinates so that $l = V(X_2, X_3)$, so that the points of $l$ are $[a : b : 0 : 0]$. In this case the coefficients of $X^3$, $X^2Y$, $XY^2$ and $Y^3$ must all vanish. The fibre is a copy of a linear subspace of dimension 15 in $\mathbb{P}^{19}$.

Aliter: Pick four distinct points $p_1$, $p_2$, $p_3$ and $p_4$ of $l$. Suppose $F(p_i) = 0$, for $1 \leq i \leq 4$. Then $F|_l$ is a cubic polynomial in two variables, vanishing at four points. Thus $F|_l$ is the zero polynomial. It follows that $l \subset V(F)$ if and only if $F$ vanishes at $p_i$, for $1 \leq i \leq 4$.

The condition that $F(p_i) = 0$ imposes one linear constraint. One can check that these four points impose independent conditions, so that the space of cubics containing all four points is a linear subspace of dimension 15.
Either way, $\Sigma$ fibres over an irreducible base with irreducible fibres of the same dimension. It follows that $\Sigma$ is irreducible of dimension $4 + 15 = 19$. It suffices then to exhibit a single cubic with finitely many lines, since then the morphism $g$ must be dominant, whence surjective. It is a fun exercise to compute the twenty seven lines on $X^3 + Y^3 + Z^3 + T^3 = 0$. □

Lemma 15.7. If $S \subset \mathbb{P}^3$ is a smooth cubic surface and $l \subset S$ is a line then there are ten lines meeting $l$.

In particular $S$ contains two skew lines.

Proof. Consider the planes $H \subset \mathbb{P}^3$ containing $l$. Then $H \cap S = l \cup C$, where $C \subset H \simeq \mathbb{P}^2$ is a plane curve of degree two.

First observe that $C$ is never a double line $n$. Indeed, if $F$ and $G$ are the linear polynomials which define $l$ and $F$ and $H$ are the linear polynomials defining $n$, so that $F = 0$ is the plane spanned by $l$ and $n$, then the equation of $S$ has the form

$$FQ + GH^2,$$

for some quadratic polynomial $Q$. But then $S$ is singular at the two points where $F = H = Q = 0$ (just compute partials).

Suppose that $m$ is a line that intersects $l$. Then $C = m \cup n$, where $n$ is another line, which also meets $l$. Thus lines that intersect $l$ come in concurrent pairs and we just have to show that there are five such pairs.

We may suppose that $l$ is given by $Z = T = 0$. Then $S$ is defined by an equation of the form

$$AX^2 + 2BXY + CY^2 + 2DX + 2EY + F,$$

where $A$, $B$, $C$, $D$, $E$ and $F$ are homogeneous polynomials in $Z$ and $T$.

The pencil of planes containing $l$ is given by $Z = \lambda T$. Note that $C = C_\lambda$ is a pair of lines if and only if $C$ is singular. $C_\lambda$ is singular if and only if the determinant

$$\begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix}$$

is zero. The determinant is a homogeneous polynomial of degree 5 in $Z$ and $T$ and so it suffices to show it has no repeated roots.

Suppose that $Z = 0$ is a root. There are two cases. If the singular point $s$ of $C_0$ is not a point of $l$ then we may assume that $C_0$ is given by $XY = 0$. Then every entry of the matrix above is divisible by $Z$. 


except $B$. On the other hand, as $s$ is not a singular point of $S$ it follows that $Z^2$ does not divide $F$. Thus $Z^2$ does not divide the determinant. If $s$ is a point of $l$ then we may assume that $C_0$ is given by $X^2 - T^2 = 0$ and one can check that $Z^2$ does not divide the determinant. \hfill \qed

Proof of (15.4). We just prove that $S$ contains a pair of skew lines. (15.6) implies that $S$ contains at least one line $l$. (15.7) implies that there are ten other lines meeting $l$. Pick one of them $l'$. Of the ten lines meeting $l'$, at most one of them intersects $l$. Thus we may find a line $m$ meeting $l'$ not intersecting $l$. \hfill \qed

Proof of (15.3). By assumption $S$ contains two skew lines $l$ and $m$. Define a rational map

$$\phi: l \times m \dashrightarrow S,$$

by sending the point $(p, q)$ to the intersection of the line $n = \langle p, q \rangle$ with $S - (l \cup m)$. Since a cubic intersects a typical line in three points, and the line $n$ intersects $S$ at $p \in l$ and $q \in m$, there is an open subset of $l \times m$ such that the line $n$ intersects $S$ at one further point $r = \phi(p, q)$.

Define a rational map

$$\psi: S \dashrightarrow l \times m,$$

by sending $r \in S - (l \cup m)$ to $(p, q)$, where $p$ is the intersection point of the plane $\langle p, m \rangle$ with $l$ and $q$ is the intersection point of the plane $\langle p, l \rangle$ with $m$.

It is easy to check that $\phi$ and $\psi$ are inverse. It follows that $\phi$ is birational. As $\mathbb{P}^1 \times \mathbb{P}^1 \simeq l \times m$ is rational, $S$ is rational. \hfill \qed