2. Rational maps

It is often the case that we are given a variety X and a morphism defined on an open subset U of X. As open sets in the Zariski topology are very large, it is natural to view this as a map on the whole of X, which is not everywhere defined.

Definition 2.1. *A* ***rational map*** $\phi\colon X \dashrightarrow Y$ *between quasi-projective varieties is a pair* (f, U) *where* U *is a dense open subset of* X *and* $f\colon U \longrightarrow Y$ *is a morphism of varieties. Two rational maps* (f_1, U_1) *and* (f_2, U_2) *are considered equal if there is a dense open subset* $V \subset U_1 \cap U_2$ *such that the two functions* $f_1|_V$ *and* $f_2|_V$ *are equal.*

It is customary to avoid using the pair notation and to leave U unspecified. We often say in this case that ϕ is defined on U. Note that if U and V are two dense open sets, and (f, U), (g, V) represent the same rational map, then $(h, U \cup V)$ also represents the same map, where h is defined in the obvious way. By Noetherian induction, it follows that there is a largest open set on which ϕ is defined, which is called the **domain of** ϕ. The complement of the domain is called the **locus of indeterminancy**.

One way to get a picture of a rational map, is to consider the graph.

Definition 2.2. *Let* $\phi\colon X \dashrightarrow Y$ *be a rational map.*

The ***graph of*** ϕ *is the closure of the graph of* f*, where the pair* (f, U) *represents* ϕ*.*

The ***image of*** ϕ *is the image of the graph of* ϕ *under the second projection.*

Note that the domain of ϕ is precisely the locus where the first projection map is an isomorphism.

Definition 2.3. *Let* $\phi\colon X \dashrightarrow Y$ *and* $\psi\colon Y \dashrightarrow Z$ *be two rational maps. Suppose that* $\phi = (f, U)$ *and* $\psi = (g, V)$ *and that* $f^{-1}(V)$ *is dense (if* X *is irreducible this is equivalent to the requirement that* $f(U) \cap V$ *is non-empty). Then we may define the composition of* ϕ *and* ψ *by taking the pair* $(g \circ f, f^{-1}(V))$*.*

Note that in general, we cannot compose rational maps. The problem might be that the image of the first map might lie in the locus where the second map is not defined. However there will never be a problem if X is irreducible and ϕ is dominant:

Definition 2.4. *We say that* ϕ *is* ***dominant*** *if the the image of* ϕ *is dense in* Y*.*

Note that this gives us a category, the category of irreducible varieties and dominant rational maps.

Definition 2.5. *We say that a dominant rational map* $\phi\colon X \dashrightarrow Y$ *of irreducible quasi-projective varieties is birational if it has an inverse. In this case we say that* X *and* Y *are* ***birational****. We say that* X *is* ***rational*** *if it is birational to* $\mathbb{P}^n$*.*

It is interesting to see an example. Let $\phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the map

$$[X : Y : Z] \longrightarrow [YZ : XZ : XY].$$

This map is clearly a rational map. It is called a **Cremona transformation**. Note that it is a priori not defined at those points where two coordinates vanish. To get a better understanding of this map, it is convenient to rewrite it as

$$[X : Y : Z] \longrightarrow [1/X : 1/Y : 1/Z].$$

Written as such it is clear that this map is an involution, so that it is in particular a birational map.

It is interesting to check whether or not this map really is well defined at the points $[0 : 0 : 1]$, $[0 : 1 : 0]$ and $[1 : 0 : 0]$. To do this, we need to look at the graph.

Consider the following map,

$$\mathbb{A}^2 \dashrightarrow \mathbb{A}^1,$$

which assigns to a point $p \in \mathbb{A}^2$ the slope of the line connecting the point p to the origin,

$$(x, y) \longrightarrow y/x.$$

Now this map is not defined along the locus where $x = 0$. Replacing $\mathbb{A}^1$ with $\mathbb{P}^1$ we get a map

$$(x, y) \longrightarrow [x : y].$$

Now the only point where this map is not defined is the origin. We consider the graph,

$$\Gamma \subset \mathbb{A}^2 \times \mathbb{P}^1.$$

Consider how the graph sits over $\mathbb{A}^2$. Outside the origin, the first projection is an isomorphism. Over the origin, the graph is contained in a copy of the image, that is, $\mathbb{P}^1$. Consider any line $y = tx$, through the origin. Then this line, minus the origin, is sent to the point with slope t. It follows that the closure of this line is sent to the point with slope t. Varying t, it follows that any point of the fibre over $\mathbb{P}^1$ is a point of the graph.

Thus the morphism $p\colon \Gamma \longrightarrow \mathbb{A}^2$ is an isomorphism outside the origin and contracts a whole copy of $\mathbb{P}^1$ to a point. For this reason, we call p a blow up.

Definition 2.6. *Let $\phi\colon X \dashrightarrow \mathbb{P}^k$ be a rational map, which is given locally by $f_1, f_2, \dots, f_k$. Let I be the ideal spanned by $f_1, f_2, \dots, f_k$. The induced morphism $p\colon \Gamma \longrightarrow X$ is called the* ***blow up of the ideal*** *I.*

Clearly p is always birational, as it is an isomorphism outside $V(I)$.

In our case $I = \langle x, y\rangle$, the maximal ideal of p, so that we call p the blow up of a point. Suppose we have coordinates $[S : T]$ on $\mathbb{P}^1$. Then outside of the origin, the graph satisfies the equation $xT = yS$. Thus the closure must satisfy the same equation. Since this equation determines the graph outside the origin, in fact the graph is defined by this equation (as the whole fibre over the origin lives in the graph, we don't need anymore equations).

The inverse image of the origin is called the **exceptional divisor**.

Definition 2.7. *Let $\pi\colon X \longrightarrow Y$ be a birational morphism. The locus where π is not an isomorphism is called the* ***exceptional locus****. If $V \subset Y$, the inverse image of V is called the* ***total transform****. Let Z be the image of the exceptional locus. Suppose that V is not contained in Z. The* ***strict transform of*** *V is the closure of the inverse image of $V - Z$.*

It is interesting to compute the strict transform of some planar curves. We have already seen that lines through the origin lift to curves that sweep out the exceptional divisor. In fact the blow up separates the lines through the origin. These are then the fibres of the second morphism.

Let us now take a nodal cubic,

$$y^2 = x^2 + x^3.$$

We want to figure out its strict transform, so that we need the inverse image in the blow up. Outside the origin, there are two equations to be satisfied,

$$y^2 = x^2 + x^3 \qquad \text{and} \qquad xT = yS.$$

Passing to the coordinate patch $y = xt$, where $t = T/S$, and substituting for y in the first equation we get

$$x^2t^2 - x^2 - x^3 = x^2(t^2 - x - 1).$$

Now if $x = 0$, then $y = 0$, so that in fact locally $x = 0$ is the equation of the exceptional divisor. So the first factor just corresponds to the exceptional divisor. The second factor will tell us what the closure of our curve looks like, that is, the strict transform. Now over the origin, $x = 0$, so that $t^2 = 1$ and $t = \pm 1$. Thus our curve lifts to a curve which intersects the exceptional divisor in two points. (If we compute in the coordinate patch $x = sy$, we will see that the curve does not meet the point at infinity). These two points correspond to the fact that the nodal cubic has two tangent lines at the origin, one of slope 1 and one of slope -1.

Now consider what happens for the cuspidal cubic, $y^2 = x^3$. In this case we get

$$(xt)^2 - x^3 = x^2(t^2 - x).$$

Once again the factor of x^2 corresponds to the fact that the inverse image surely contains the exceptional divisor. But now we get the equation $t^2 = 0$, so that there is only one point over the origin, as one might expect from the geometry.

Let us go back to the Cremona transformation. To compute what gets blown up and blown down, it suffices to figure out what gets blown down, by symmetry. Consider the line $X = 0$. If $bc \neq 0$, the point $[0 : b : c]$ gets mapped to $[0 : 0 : 1]$. Thus the strict transform of the line $X = 0$ in the graph gets blown down to a point. By symmetry the strict transforms of the other two lines are also blown down to points. Outside of the union of these three lines, the map is clearly an isomorphism.

Thus the involution blows up the three points $[0 : 0 : 1]$, $[0 : 1 : 0]$, and $[1 : 0 : 0]$ and then blows down the three disjoint lines. Note that the three exceptional divisors become the three new coordinate lines.

One of the most impressive results of the nineteenth century is the following characterisation of the birational automorphism group of $\mathbb{P}^2$.

Theorem 2.8 (Noether)**.** *The birational automorphism group is generated by a Cremona transformation and* PGL(3)*.*

This result is very deceptive, since it is known that the birational automorphism group is, by any standards, very large.