5. Some naive enumerative geometry

Question 5.1. *How many lines meet four fixed lines in $\mathbb{P}^3$?*

Let us first check that this question makes sense, that is, let us first check that the answer is finite.

Definition 5.2. $\mathbb{G}(k, n)$ *denotes the space of r-dimensional linear subspaces of $\mathbb{P}^n$.*

We will assume that we have constructed the **Grassmannian** as a variety. The first natural question then is to determine the dimension of $\mathbb{G}(1, 3)$. We do so in an ad hoc manner. A line l in $\mathbb{P}^3$ is specified by picking two points p and q. Now the set of choices for two points p and q is equal to $\mathbb{P}^3 \times \mathbb{P}^3 - \Delta$, where Δ is the diagonal. Thus the set of choices of pairs of distint points is six dimensional.

Fix a line l. Then if we pick any two points p and q of this line, they give us the same line l. Thus the the Grassmannian of lines in $\mathbb{P}^3$ is $6 = 2 = 4$-dimensional.

It might help to look at this differently. Let

$$\Sigma = \{ (P, l) \mid P \in l \} \subset \mathbb{G}(1, 3) \times \mathbb{P}^3.$$

Then Σ is a closed subset of the product $\mathbb{G}(1, 3) \times \mathbb{P}^3$. There are two natural projection maps.

$$\begin{tikzcd}
\Sigma \arrow[r, "q"] \arrow[d, "p"'] & \mathbb{P}^3 \\
\mathbb{G}(1,3). &
\end{tikzcd}$$

In fact Σ (together with this diagram) is called an **incidence correspondence**. It is interesting to consider the two morphisms p and q. First p. Pick a line $l \in \mathbb{G}(1, 3)$. Then the fibre of p over l consists of all points P that are contained in l, so that the fibres of p are all isomorphic to $\mathbb{P}^1$. Now consider the morphism q. Fix a point P. Then the fibre of q over P consists of all lines that contain P. Again the fibres of q are isomorphic.

To compute the dimension of $\mathbb{G}(1, 3)$, we compute the dimension of Σ in two ways, borrowing the following result from later in the class.

Theorem 5.3. *Let $\pi \colon X \longrightarrow Y$ be a dominant morphism of irreducible varieties.*

Then there is an open subset U of Y, such that for every $y \in U$, the dimension of the fibre of π over is equal to k, a constant. Moreover the dimension of X is equal to the dimension of Y plus k.

We first apply (5.3) to q. The dimension of the base is 3. As every fibre is isomorphic, to compute k, we can consider any fibre. Pick any point P. Pick an auxiliary plane, not passing through P. Then the set of lines containing P is in bijection with the points of this auxiliary plane, so that the dimension of a fibre is two. Thus the dimension of $\Sigma = 3 + 2 = 5$.

Now we apply (5.3) to p. The dimension of any fibre is one. Thus the dimension of the Grassmannian is $5 - 1 = 4$, as before.

The next question is to determine how many conditions it is to meet a fixed line l_1. I claim it is one condition. The easiest way to see this, is to just to imagine swinging a sword around in space. This will cut any line into two. Thus any one dimensional family of lines meets a given line finitely many times.

More formally, carry out the computation above, replacing Σ with Σ_1, the space of lines which meet l_1. The fibre of q over a point P is now a copy of $\mathbb{P}^1$ (parametrised by l_1). Thus Σ_1 has dimension 4 and the space of lines which meets l_1 has dimension $4 - 1 = 3$.

Thus we have a threefold in the fourfold $\mathbb{G}(1,3)$. Clearly we expect that four of them will intersect in a finite set of points.

There are two ways to proceed. Here is one which uses the Segre variety:

Lemma 5.4. *Let l_1, l_2 and l_3 and m_1, m_2 and m_3 be two sequences of skew lines in $\mathbb{P}^3$.*

Then there is an element of $\mathrm{PGL}_4(K)$ carrying the first sequence to the second.

Proof. We may as well assume that the first set is given as

$$X = Y = 0 \qquad\qquad Z = W = 0 \qquad \text{and} \qquad X - Z = Y - W = 0.$$

Clearly we may find a transformation carrying m_1 to l_1 and m_2 to l_2. For example, pick four points on both sets of lines, and use the fact that any four sets of points in linear general position are projectively equivalent.

Consider the two planes $X = 0$ and $W = 0$. m_3 cannot lie in either of these planes, else either the lines m_1 and m_3 or the lines m_2 and m_3 would not be skew. Consider the two points $[0 : a : b : c]$ and $[d : e : f : 0]$ where m_3 intersects the planes $X = 0$ and $W = 0$.

Clearly m_3 is determined by these points, and in the case of l_3, we have $a = c = d = f = 1$, $b = e = 0$. Pick an element $\phi \in \mathrm{PGL}_4(K)$ and represent it as a 4×4 matrix. If we decompose this 4×4 matrix

in block form
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$
where each block is a 2×2 matrix, the condition that ϕ fix $l_1 = m_1$ and $l_2 = m_2$ is equivalent to the condition that $B = C = 0$. By choosing A and D appropriately, we reduce to the case that $b = e = 0$. In this case the matrix
$$\begin{pmatrix} 1/d & 0 & 0 & 0 \\ 0 & 1/a & 0 & 0 \\ 0 & 0 & 1/f & 0 \\ 0 & 0 & 0 & 1/c \end{pmatrix}$$
carries the two points to the standard two points, so that it carries m_3 to l_3. $\square$

This result has the following surprising consequence.

Lemma 5.5. *Let l_1, l_2 and l_3 be three skew lines in $\mathbb{P}^3$.*

Then the family of lines that meets all three lines sweeps out a quadric surface in $\mathbb{P}^3$.

Proof. By (5.4) we may assume that the three lines are any set of three skew lines in $\mathbb{P}^3$. Now the Segre variety V in $\mathbb{P}^1 \times \mathbb{P}^1$ contains three skew lines (just choose any three lines of one of the rulings). Moreover any line of the other ruling certainly meets all three lines. So the set of lines meeting all three lines, certainly sweeps out at least a quadric surface.

To finish, suppose we are given a line l that meets l_1, l_2 and l_3. Then l meets V in three points. As V is defined by a quadratic polynomial, it follows that l is contained in V. Thus any line that meets all three lines, is contained in V. $\square$

Theorem 5.6. *There are two lines that meet four general skew lines in $\mathbb{P}^3$.*

Proof. Fix the first three lines l_1, l_2 and l_3. We have already seen that the set l of lines that meets all three of these lines is precisely the set of lines of one ruling of the Segre variety (up to choice of coordinates).

Pick a line l_4 that meets V transversally in two points. Now for a line l of one ruling to meet the fourth line l_4, it must meet l_4 at a point $P = l \cap l_4$ of V. Moreover this point determines the line l. $\square$

Here is an entirely different way to answer (5.1). Consider using the principle of continuity. Take two of the four lines and deform them so they become concurrent (or what comes to exactly the same thing, coplanar). Similarly take the other pair of lines and degenerate them until they also become concurrent.

Now consider how a line l can meet the four given lines.

Lemma 5.7. *Let l be a line that meets two concurrent lines l_1 and l_2 in $\mathbb{P}^3$.*

Then either l contains $l_1 \cap l_2$ or l is contained in the plane $\langle l_1, l_2 \rangle$.

Proof. Suppose that l does not contain the point $l_1 \cap l_2$. Then l meets l_i, $i = 1, 2$ at two points p_i contained in the plane $\langle l_1, l_2 \rangle$. □

Thus if l meets all four lines, there are three possibilties.

(1) l contains both points of intersection.
(2) l is contained in both planes.
(3) l contains one point and is contained in the other plane.

Clearly there is only one line that satisfies (1). It is not so hard to see that there is only one line that satisfies (2), it is the intersection of the two planes. Finally it is not so hard to see that (3) is impossible. Just choose the point outside of the plane.

Thus the answer is two. It is convenient to introduce some notation to compute these numbers, which is known as Schubert calculus. Let l denote the condition that we meet a fixed line. We want to compute l^4. We proceed formally. We have already seen that

$$l^2 = l_p + l_\pi$$

where l_p denotes the condition that a line contains a point, and l_π is the condition that l is contained in π.

Thus

$$\begin{aligned} l^4 &= (l^2)^2 \\ &= (l_p + l_\pi)^2 \\ &= l_p^2 + 2 l_p l_\pi + l_\pi^2 \\ &= 1 + 2 \cdot 0 + 1 = 2, \end{aligned}$$

where the last line is computed as before.