7. The universal family

As with the space of conics in $\mathbb{P}^2$, the main point of the Grassmannian, is that it comes with a universal family. We first investigate what this means in the baby case of quasi-projective varieties before we move on to the more interesting case of schemes.

Definition 7.1. *A **family of k-planes in $\mathbb{P}^n$ over** B is a closed subset $\Sigma \subset B \times \mathbb{P}^n$ such that the fibres, under projection to the first factor, are identified with k-planes in $\mathbb{P}^n$.*

Definition 7.2. *Let F be the functor from the category of varieties to the category of sets, which assigns to every variety, the set of all (flat) families of k-planes in $\mathbb{P}^n$, up to isomorphism.*

Theorem 7.3. *The Grassmannian $\mathbb{G}(k, n)$ represents the functor F.*

It might help to unravel some of the definitions. Suppose that we are given a variety B. Essentially we have to show that there is a natural bijection of sets,

$$F(B) = \mathrm{Hom}(B, \mathbb{G}(k, n)).$$

The set on the left is nothing more than the set of all families of k-planes in $\mathbb{P}^n$, with base B. In particular given a morphism $f : B \longrightarrow \mathbb{G}(k, n)$, we are supposed to produce a family of k-planes over B. Here is how we do this. Suppose that we have constructed the natural family of k-planes over $\mathbb{G}(k, n)$,

$$\begin{array}{ccc} \Sigma & \hookrightarrow & \mathbb{G}(k,n) \times \mathbb{P}^n \\ & & \downarrow \\ & & \mathbb{G}(k,n), \end{array}$$

so that the fibre over $[\Lambda] \in \mathbb{G}(k, n)$ is exactly the set,

$$\{[\Lambda]\} \times \Lambda \subset \{[\Lambda]\} \times \mathbb{P}^n$$

that is, the k-plane Λ sitting inside $\mathbb{P}^n$. Then we obtain a family of k-planes over B, simply by taking the fibre square,

$$\begin{array}{ccc} \Sigma' & \longrightarrow & \Sigma \\ \downarrow & \lrcorner & \downarrow \\ B & \xrightarrow{f} & \mathbb{G}(k,n). \end{array}$$

For this reason, we call the family $\Sigma \longrightarrow \mathbb{G}(k, n)$ the universal family. Note that we can reverse this process. Suppose that $\mathbb{G}(k, n)$ represents the functor F. By considering the identity morphism $\mathbb{G}(k, n) \longrightarrow \mathbb{G}(k, n)$, we get a family $\Sigma \longrightarrow \mathbb{G}(k, n)$, which is universal, in the sense that to obtain any other family, over any other base, we simply pullback Σ along the morphism $f\colon B \longrightarrow \mathbb{G}(k,n)$, whose existence is guaranteed by the universal property of $\mathbb{G}(k,n)$ (that is, that it represents the functor). To summarise the previous discussion: to prove (7.3) it suffices to construct the natural family over $\mathbb{G}(k,n)$ and prove that it is the universal family.