MODEL ANSWERS TO HWK #2

2.14. (a) We first show that the intersection of the homogeneous prime ideals is the set of nilpotent elements of S. Indeed, the intersection of the homogeneous prime ideals certainly contains all the nilpotent elements. Suppose $s \in S$ is not nilpotent. It remains to find a homogeneous prime ideal which does not contain s. As the ideal generated by the nilpotent elements is homogeneous we may assume that s is homogeneous. Pick a maximal homogeneous ideal $\mathfrak{p}$ which does not contain s. Then $\mathfrak{p}$ is a homogeneous prime ideal which does not contain s.
Now $\operatorname{Proj} S$ is empty if and only if every homogeneous prime ideal contains S_+. So $\operatorname{Proj} S$ is empty if and only if every element of S_+ is nilpotent.
(b) Let $\mathfrak{a}$ be the homogeneous ideal generated by $\phi(S_+)$. Then $U = \operatorname{Proj} T - V(\mathfrak{a})$ and so U is open. If $g \in S$ is homogeneous then $\phi\colon S \longrightarrow T$ induces a ring homomorphism $\phi_{(g)}\colon S_{(g)} \longrightarrow T_{(\phi(g))}$. This defines a morphism $\operatorname{Spec} T_{(\phi(g))} \longrightarrow \operatorname{Spec} S_{(g)}$ whence, by composition, a morphism $\operatorname{Spec} T_{(\phi(g))} \longrightarrow \operatorname{Proj} S$. On the other hand, the sets $\operatorname{Proj} T - V(\phi(g))$ form an open cover of U. As these morphisms are clearly compatible on overlaps, this induces a morphism

$$f\colon U \longrightarrow X = \operatorname{Proj} S.$$

(c) Suppose that $\mathfrak{p}$ is a homogeneous prime ideal which contains $\phi(S_+)$. Then $\mathfrak{p}$ contains T_d, for all $d \geq d_0$. Suppose that $g \in T_d$, $d \geq 1$. Then $g^k \in T_{kd}$ and for k large enough $g^k \in \mathfrak{p}$. But then $g \in \mathfrak{p}$ and $\mathfrak{p} \supset T_+$. So $U = \operatorname{Proj} T$.
Suppose that $g \in S$ is homogeneous of degree $d \geq d_0$. Consider the ring homorphism:

$$\phi_{(g)}\colon S_{(g)} \longrightarrow T_{(\phi(g))}.$$

Let $h = \phi(g)$. Suppose that $b/h^k \in T_{(h)}$. Then $b \in T_{dk}$. Pick $a \in S_{dk}$ such that $\phi(a) = b$. Then $\phi_{(g)}(a/g^k) = b/h^k$ and so $\phi_{(g)}$ is surjective. Suppose that a/g^k maps to zero, for some $k > 0$. Then $h^l\phi(a) = 0$, in $T_{(k+l)d}$ and it follows that $g^l a = 0$ in $S_{(k+l)d}$. Thus $\phi_{(g)}$ is a ring isomorphism.
Now suppose that g is any homogeneous element of S. Then g^k is also homogeneous and if k is sufficiently large then g^k has degree at least d_0, and $V(g) = V(g^k)$. Thus open sets of the form $\operatorname{Proj} S - V(g)$ and $\operatorname{Proj} T - V(g)$ cover $\operatorname{Proj} S$ and $\operatorname{Proj} T$, where g has degree at least d_0. It follows that f is an isomorphism.

It remains to find an example of this phenomena. Let

$$S = k[X, Y]/\langle X^2, XY, Y^2\rangle$$

and let $T = k[X, Y]/\langle X, Y\rangle$. Then there is a natural ring homomorphism

$$\phi \colon S \longrightarrow T.$$

This map is not an isomorphism but ϕ_d is a isomorphism of vector spaces unless $d = 1$ (indeed it is the zero map between vector spaces of dimension zero, as soon as $d \geq 2$). In fact more generally take any projective variety $X \subset \mathbb{P}^n$, let $J = I(X)$ be the homogeneous ideal of X and let I be any ideal which cuts out X scheme theoretically. Let $R = k[X_0, X_1, \dots, X_n]$, $S = R/J$ and $T = R/I$.
(d) Suppose that $V \subset \mathbb{P}^n$. Then $V_i = V \cap U_i$ forms an open affine cover of V, where U_i is the standard affine open subset of $\mathbb{P}^n$. Then $t(U_i)$ forms an open cover of V. We have already seen that $t(U_i) = \operatorname{Spec} A_i$, where A_i is the coordinate ring of V_i. But $A_i = S_{(X_i)}$. It follows that there is a natural isomorphism

$$f_i' \colon t(U_i) \longrightarrow \operatorname{Proj}(S) - V(X_i),$$

and by composition we get a morphism,

$$f_i \colon t(U_i) \longrightarrow \operatorname{Proj}(S).$$

As these morphisms are compatible on overlaps, we get a morphism

$$f \colon t(V) \longrightarrow \operatorname{Proj}(S).$$

Clearly we may also define a morphism

$$g \colon \operatorname{Proj}(S) \longrightarrow t(V),$$

using the same argument. As f and g are inverse morphisms, f is an isomorphism.
3.6 Let $U = \operatorname{Spec} A$ be any open affine subscheme. Then $\xi \in U$ and so ξ corresponds to a prime ideal of A, which must be the zero ideal, or else ξ would not the generic point. But then

$$\mathcal{O}_{X,\xi} \simeq A_{\langle 0\rangle} = K,$$

where K is the field of fractions of A.
3.8 We first check that if $X = \operatorname{Spec} A$ is an integral affine scheme then X is normal if and only if A is integrally closed. Let K be the field of fractions of A.
Suppose first that X is normal. Let $u \in K$ be integral over A. Let $p \in X$ be a point. Then u is integral over $\mathcal{O}_{X,p}$. As X is normal, $\mathcal{O}_{X,p}$ is integrally closed, so that $u \in \mathcal{O}_{X,p}$. As a function is regular if and

only if it is regular at every point, we have $u \in \mathcal{O}_X(X) = A$. Thus A is integrally closed.

Suppose that A is integrally closed. Let $u \in K$ be integral over $\mathcal{O}_{X,p}$. By assumption we may find $\alpha_i \in \mathcal{O}_{X,p}$ such that u is a root of the monic polynomial

$$t^n + \alpha_{n-1}t^{n-1} + \cdots + \alpha_1 t + \alpha_0.$$

p corresponds to a prime ideal $\mathfrak{p}$ of A, by definition, and

$$\mathcal{O}_{X,p} \simeq A_{\mathfrak{p}}.$$

Thus we may find $a_i \in A$ and $f_i \notin \mathfrak{p}$ such that

$$\alpha_i = \frac{a_i}{f_i}.$$

Let f be the product of $f_0, f_1, \dots, f_{n-1}$ and let $v = uf$. Multiplying through by f^n, we get

$$\begin{aligned} & v^n + f\alpha_{n-1}vf^{n-1} + \cdots + f^{n-1}\alpha_1 v + \alpha_0 \\ &= f^n u^n + f^n\alpha_{n-1}(u)^{n-1} + \cdots + f^n\alpha_1 u + \alpha_0 \\ &= 0. \end{aligned}$$

As $f^i\alpha_{n-i} \in A$, it follows that v is integral over A. As A is integrally closed, $v \in A$. But $f \notin \mathfrak{p}$ and so $u = v/f \in \mathcal{O}_{X,p}$. Therefore $\mathcal{O}_{X,p}$ is integrally closed, so that X is normal.

Now we check the patching condition. Suppose that U and V are two affine open subschemes of X. Let $\tilde{U} = \operatorname{Spec} \tilde{A}$ and $\tilde{V} = \operatorname{Spec} \tilde{B}$. We have to exhibit a canonical isomorphism

$$\phi \colon U' \longrightarrow V',$$

where U' is the inverse image of $U \cap V$ in $\tilde{U}$ and V' is the inverse image of $U \cap V$ in $\tilde{V}$.

Since it suffices to construct a canonical morphism on an open cover, we may assume that U and V are open affines of a common affine scheme $W = \operatorname{Spec} C$ and that $A = C_f$ and $B = C_g$, where f and g belong to C. It suffices to check that if $\tilde{A}$ is the integral closure of A, then $\tilde{A}_f$ is the integral closure of A_f. It is clear that any element of $\tilde{A}_f$ is integral over A_f. Indeed if $a/f^k \in \tilde{A}_f$, where $a \in \tilde{A}$ satisfies the monic polynomial

$$x^n + a_{n-1}x^n + \cdots + a_0,$$

then a/f^k satisfies the monic polynomial

$$x^n + b_{n-1}x^n + \cdots + b_0,$$

where $b_i = a_i/f^{n(k-i)}$. On the other hand if u belong to the integral closure of A_f, then u is a root of a monic polynomial

$$x^n + b_{n-1}x^n + \cdots + b_0,$$

where each $b_i \in A_f$. Clearing denominators, it follows that $a = f^l u \in \tilde{A}$, for an appropriate power of f. Thus one can glue the schemes $\tilde{U}$ together to get a scheme $\tilde{X}$. The inclusion $A \longrightarrow \tilde{A}$ induces a morphism of schemes $\tilde{U} \longrightarrow U$, whence a morphism of schemes $\tilde{U} \longrightarrow X$. Arguing as before, these morphisms agree on overlaps. It follows that there is an induced morphism $\tilde{X} \longrightarrow X$.

Now suppose that there is a dominant morphism of schemes $Z \longrightarrow X$, where Z is normal. This induces a dominant morphism $Z_U \longrightarrow U$, where U is an open affine subscheme and Z_U is the inverse image of U Thus it suffices to prove the universal property of $X = \operatorname{Spec} A$ in the case when X is affine. Covering Z by open affines, it suffices to prove this result when $Z = \operatorname{Spec} B$ is affine.

Note that as $Z \longrightarrow X$ is dominant then the induced ring homomorphism $A \longrightarrow B$ is injective. Let L be the function field of Z so that L is the field of fractions of B. Then there is an induced field homomorphism $K \longrightarrow L$. If $\tilde{X} = \operatorname{Spec} \tilde{A}$, so that $\tilde{A}$ is the integral closure of A, then $A \subset \tilde{A} \subset K$ and there is an induced ring homomorphism $\tilde{A} \longrightarrow L$. As Z is normal, B is integrally closed. On the other hand, any element of the image is obviously integral over the image of A, and so integral over B. But then the image of $\tilde{A}$ lies in B, as B is integrally closed. This induces a natural morphism $Z \longrightarrow \tilde{X}$, which factors $\tilde{X} \longrightarrow X$.

Suppose that X is of finite type. Clearly we may assume that $X = \operatorname{Spec} A$ is affine. We are reducing to showing that the integral closure $\tilde{A}$ of a finitely generated k-algebra A, is a finitely generated A-module. But this is a well-known result in algebra.

3.9 (a)

$$\mathbb{A}^2_k = \operatorname{Spec} k[x,y] = \operatorname{Spec}(k[x] \underset{k}{\otimes} k[y]) = \mathbb{A}^1_k \underset{k}{\times} \mathbb{A}^1_k.$$

The points of $\mathbb{A}^1_k$ consist of the maximal ideals $\mathfrak{m}_a$ and the generic point ξ. The points of the product of sets are then ordered pairs $(\mathfrak{m}_a, \mathfrak{m}_b)$, with closure $\{(\mathfrak{m}_a, \mathfrak{m}_b)\}$, $(\mathfrak{m}_a, \xi)$, with closure

$$\{\, (\mathfrak{m}_a, \mathfrak{m}_b) \mid b \in k \,\} \cup \{(\mathfrak{m}_a, \xi)\},$$

$(\xi, \mathfrak{m}_b)$ with closure

$$\{\, (\mathfrak{m}_a, \mathfrak{m}_b) \mid a \in k \,\} \cup \{(\xi, \mathfrak{m}_b)\},$$

and (ξ, ξ), whose closure is the whole space. Let $\eta = \langle xy - 1 \rangle$. Then η is a prime ideal, whose closure is the set

$$\{ (\mathfrak{m}_a, \mathfrak{m}_b) \mid ab = 1 \} \cup \{\eta\}.$$

Thus η is not a point of the product of the two sets.
(b) Let $X = \operatorname{Spec} k(s) \underset{k}{\times} k(t)$. As there is a morphism

$$\operatorname{Spec} k(t) \longrightarrow \operatorname{Spec} k[t] \qquad \text{induced by} \qquad k[t] \longrightarrow k(t),$$

there is a morphism

$$X \longrightarrow \mathbb{A}^2_k.$$

Let U be the image. $k(s) \underset{k}{\otimes} k(t)$ is the localisation of $k[s,t]$ of the multiplicative set S generated by the irreducible polynomials in s and t. It follows that X is isomorphic to U. U is obtained by throwing out every closed point, and every line parallel to either axis. Equivalently the points of U the generic point $\mathbb{A}^2_k$ and the generic point of every irreducible curve except the x-axis or y-axis.
3.11 (a) We first check this in the special case when $X' \longrightarrow X$ is an open immersion. In this case the image of Y' is clearly closed, the restricted morphism is a homeomorphism and surjectivity of $\mathcal{O}_{X'} \longrightarrow f_*\mathcal{O}_{Y'}$ is clear. In particular, it is easy to deduce that f is a closed immersion if and only if there is a cover by open immersions $X' \longrightarrow X$ (meaning simply that X is the union of the images) such that f' is a closed immersion, for every open set of the cover.
So to check the general case, we may assume that $X = \operatorname{Spec} A$ is affine. Let $V \subset Y$ be an open affine subset of Y. We may find an open subset $U \subset X$ such that $f^{-1}(U) = V$. Then we may find a regular function f on X (or better $f \in A$) such that $U_f \subset U$. Then $f^{-1}(U_f)$ is an open affine subset of V. Since U_f cover U, we may assume that X and $Y = \operatorname{Spec} B$ are both affine. In this case B is a quotient of A. Finally we may assume that $X' = \operatorname{Spec} A'$ is affine. Since $B' = B \underset{A}{\otimes} A'$ is a quotient of A', f' is indeed a closed immersion.
(b) Pick an open affine cover $\{Y_\alpha\}$ of Y. Then there is an open subset X_α of X such that $Y_\alpha = Y \cap X_\alpha$. We may find f_i such that for every α there is an index i such that $U_{f_i} \subset X_\alpha$. Then $U_{f_i} \cap Y$ is an open affine subset of Y, as it is equal to the locus where the regular function $f|Y_\alpha$ is not zero on the affine scheme Y_α. By compactness we may assume there are only finitely many $f_1, f_2, \dots, f_r$. $f_1, f_2, \dots, f_r$ generate the unit ideal as the sets U_{f_i} are an open affine cover of X. By (2.17.b) Y is affine. Now apply (2.18.d).
(c) We want to give a morphism of schemes $Y \longrightarrow Y'$. The map on topological spaces is simply the identity. Pick an open affine cover of

X. By part (b) this induces an open affine cover of Y and Y'. On this affine cover if Y and Y' are given by ideals $\mathfrak{a}$ and $\mathfrak{a}'$ in the ring A, then $\mathfrak{a}$ is the radical of $\mathfrak{a}'$. In particular there is a natural inclusion $\mathfrak{a} \subset \mathfrak{a}'$ and so a natural surjection $A/\mathfrak{a}' \longrightarrow A/\mathfrak{a}$ which factors $A \longrightarrow A/\mathfrak{a}'$ and $A \longrightarrow A/\mathfrak{a}$. This gives us a commutative diagram

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ & \searrow & \downarrow \\ & & X. \end{array}$$

These maps automatically glue, by naturality.
(d) We first suppose that $X = \operatorname{Spec} A$ is affine. In this case there is a homomorphism of rings,

$$A \longrightarrow H^0(Z, \mathcal{O}_Z).$$

Let $\mathfrak{p}$ be the kernel and let B be the quotient, so that there is a ring commutative diagram,

$$\begin{array}{ccc} A & \longrightarrow & B \\ & \searrow & \downarrow \\ & & H^0(Z, \mathcal{O}_Z). \end{array}$$

Let $Y = \operatorname{Spec} B$. Then, Y is a closed subscheme of X and there is a commutative diagram

$$\begin{array}{ccc} X & \longleftarrow & Y \\ & \nwarrow & \uparrow \\ & & Z. \end{array}$$

Now suppose that there is another commutative diagram,

$$\begin{array}{ccc} X & \longleftarrow & Y' \\ & \nwarrow & \uparrow \\ & & Z. \end{array}$$

Then there is an induced map of rings,

$$\begin{array}{ccc} A & \longrightarrow & H^0(Y', \mathcal{O}_{Y'}). \\ & \searrow & \downarrow \\ & & H^0(Z, \mathcal{O}_Z). \end{array}$$

By the universal property of the quotient, there is an induced ring homomorphism,

$$H^0(Y', \mathcal{O}_{Y'}) \longrightarrow B,$$

and this gives rise to a morphism of schemes $Y \longrightarrow Y'$.
Now suppose that X is arbitrary. Pick an open affine cover $\{U_i\}$ of X, such that U_{ij} is affine. Let V_i be the inverse image of U_i in X_i. Let $g_i \colon Y_i \longrightarrow X$ be the affine scheme constructed above. Let $Y_{ij} = g_i^{-1}(U_j)$ be the inverse image of U_j. Then Y_{ij} and Y_{ji} satisfy the same universal property and so there are induced isomorphisms ϕ_{ij} which satisfy the cocycle condition. Glueing together the Y_i, this defines Y. Y is a closed subscheme of X and it clearly satisfies the given universal property.
The last property is clear, since both Y and the reduced induced subscheme enjoy the same universal property.
3.12. (a) $\phi(S_+) = \phi(T_+)$, as ϕ is surjective, and so $U = \operatorname{Proj} T$. Now suppose that $g \in T$ is homogeneous. If $h = \phi(g) \in S$ then

$$\phi_{(g)} \colon S_{(h)} \longrightarrow T_{(g)},$$

is surjective. Therefore

$$f_{(g)} \colon \operatorname{Proj} T - V(h) = \operatorname{Spec} T_{(h)} \longrightarrow \operatorname{Proj} S - V(g) = \operatorname{Spec} S_{(g)},$$

is a closed immersion. As open sets of the form $\operatorname{Proj} S - V(g)$ cover $\operatorname{Proj} S$ it follows that f is a closed immersion.
(b) We have surjective ring homomorphisms $S \longrightarrow S/I'$, $S \longrightarrow S/I$ and $S/I' \longrightarrow S/I$. This gives rise to closed immersions $i\colon \operatorname{Proj} S/I' \longrightarrow \operatorname{Proj} S$, $j\colon \operatorname{Proj} S/I \longrightarrow \operatorname{Proj} S$ and $k\colon \operatorname{Proj} S/I \longrightarrow \operatorname{Proj} S/I'$, such that $j = i \circ k$. k is an isomorphism by (2.14.c) and so i and j are equivalent closed immersions. By (2.14.d) there are plenty of examples of this phenomena.
3.13 (a) Let $f \colon X \longrightarrow Y$ be a closed immersion. Suppose that $i \colon U \longrightarrow Y$ is an open immersion, where U is affine. By (3.11.a) the map $g \colon V \longrightarrow U$ obtained by pulling back the morphism f along the morphism i is a closed immersion. As U is affine, (3.11.b) implies that V is affine as well, and the map i is induced by a quotient ring homomorphism,

$$A \longrightarrow B = A/\mathfrak{a}.$$

B is clearly a finitely generated A-algebra and so f is of finite type.
(b) Let $f \colon X \longrightarrow Y$ be an open immersion. Let $U \subset X$ be an affine open subset of X. Then $f(U)$ is an open affine subset of Y which is isomorphic to U. It follows that f is locally of finite type and as f is quasi-compact, it is of finite type.
(c) Let $f \colon X \longrightarrow Y$ and $g \colon Y \longrightarrow Z$ be two morphisms of finite type and let $h \colon X \longrightarrow Z$ be the composition. Pick an open affine subset $W = \operatorname{Spec} C$ of Z. By (3.3.b) we may find a finite open affine cover $V_i = \operatorname{Spec} B_i$ of $g^{-1}(W)$ such that B_i is a finitely generated C-algebra.

For each V_i, we may find a finite open affine cover $U_{ij} = \operatorname{Spec} A_{ij}$ of $f^{-1}(V_i)$, such that C_{ij} is a finitely generated B_i-algebra.
But then $U_{ij} = \operatorname{Spec} A_{ij}$ is a finite open affine cover of $h^{-1}(W)$ where A_{ij} is a finitely generated C-algebra. Therefore h is of finite type.
(d) Let $f \colon X \longrightarrow Y$ be a morphism of finite type and let $Y' \longrightarrow Y$ be a morphism. Let $f' \colon X' \longrightarrow Y'$ be the induced morphism, where X' is the fibre product of X and Y' over Y. We want to prove that f' is of finite type. Let $V = \operatorname{Spec} B$ be an open subset of Y. Then there is a finite open affine cover $U_i = \operatorname{Spec} A_i$ of $f^{-1}(V)$, where A_i is a finitely generated B-algebra.
(e) By part (d) $X \underset{S}{\times} Y \longrightarrow Y$ is of finite type. But then the morphism $X \underset{S}{\times} Y \longrightarrow S$ is of finite type, as it is a composition of morphisms of finite type.
(f) Let $W = \operatorname{Spec} C$ be an affine open subset of Z. By assumption $(g \circ f)^{-1}(W)$ can be covered by affine open subsets $U = \operatorname{Spec} A$ of X, where A is a finitely generated C-algebra. Pick an affine open subset $V = \operatorname{Spec} B$ of $g^{-1}(W)$. Then we can cover $f^{-1}(V) \cap U$ with affine open subsets of the form $\operatorname{Spec} A_h$, where h is a regular function on U. As A_h is a finitely generated C-algebra it is a finitely generated B-algebra. But then f is locally of finite type and as f is compact, it is of finite type.
(g) Let $V = \operatorname{Spec} B$ be an affine subset of Y. Then $f^{-1}(V)$ is a finite union of affine sets of the form $U = \operatorname{Spec} A$, where A is a finitely generated B-algebra. As B is Noetherian, A is Noetherian and so X is Noetherian.
3.15 We first make some general observations that apply to both parts (a) and (b).
Suppose that X is of finite type over a field k. Then X has a finite cover $U_i = \operatorname{Spec} A_i$ by open affines, where A_i is a finitely generated k-algebra. If U_i and U_j don't intersect then $U_i \cup U_j = \operatorname{Spec} A_i \oplus A_j$ is affine. So we may assume that $U_i \cap U_j$ is non-empty. But then X is irreducible or reduced if and only if U_i is irreducible or reduced, for all i.
If K/k is any field extension, then $Y = X \underset{\operatorname{Spec} k}{\times} \operatorname{Spec} K$ is covered by open affines of the form $V_i = \operatorname{Spec} B_i = \operatorname{Spec} A_i \underset{k}{\otimes} K$. As $U_i \cap U_j$ is non-empty, so is $V_i \cap V_j$. Thus Y is irreducible or reduced if and only if V_i is irreducible or reduced for all i.
So we might as well assume that $X = \operatorname{Spec} A$ is affine. If X is irreducible and Y is reducible, then Xred is irreducible and Yred is reducible. Similarly if X is reduced and Y is not reduced then every

irreducible component of X is reduced and some irreducible component of Y is not reduced. Hence, we may also assume that X is integral and Y is not integral. Note that the field extension K/k is the limit of the finitely generated intermediary field extensions $K/L/k$. It follows that the tensor product B is the limit of the tensor products $B_L = A \underset{k}{\otimes} L$. Thus the scheme Y is the limit of the schemes $X_L = \operatorname{Spec} B_L$. Since the limit of integral schemes is integral, we may assume that $K = L$. By induction we may assume that $K = k(\alpha)$ is primitive.
Suppose that $\alpha = x$ is transcendental over k. In this case B is a localisation of $A[x]$. $A[x]$ is clearly integral and so B is integral.
(a) It suffices to consider the case $\alpha^p \in k$. In this case $B = A[\alpha]$. Suppose that f and $g \in B$ and $fg = 0$. Now

$$f = f_0 + f_1\alpha + \cdots + f_{p-1}\alpha^{p-1}$$

for $f_0, f_1, \dots, f_{p-1} \in A$. Hence $f^p \in A$. Similarly $g^p \in A$. As $f^p g^p = 0$ and A is integral either $f^p = 0$ or $g^p = 0$. It follows that Y is irreducible.
(b) It suffices to prove that if α is separable over k then Y is reduced. Replacing K by its Galois closure, we may assume that K/k is Galois, with Galois group G. Thus G acts on B and $B^G = A$. Suppose that $f^2 = 0$. Consider

$$\phi(x) = \prod_{g \in G} (x - g^* f) \in B[x].$$

As this polynomial is invariant under G, in fact

$$\phi(x) = a_0 + a_1 x + \cdots + a_k x^k \in A.$$

As $f^2 = 0$ we have

$$0 = \phi(f) = a_0 + a_1 f.$$

Thus $f \in A$ and so $f = 0$, since A is integral. Thus B does not have any nilpotents and Y is reduced.
(c) Consider $\operatorname{Spec} k(t)[x]/\langle x^6 - t\rangle$, where $k = \mathbb{F}_2$. As $x^6 - t \in k(t)[x]$ is irreducible, this scheme is integral. But if we replace t by t^6 (equivalently, we adjoin a root of $x^6 - t$) then $x^3 - t^3$ is a non-zero global section which squares to zero and $x^3 - t^3$ factors non-trivially, so that $\operatorname{Spec} k(t)[x]/\langle x^6 - t^6\rangle$ is neither reduced nor irreducible.