Let A be an $m \times n$ matrix. There are two ways to view matrix multiplication. The first focuses on the rows of A. This is the usual way to compute and is the way that equations arise:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 3x + 4y \end{pmatrix}.$$

But there is another way to view matrix multiplication, focusing on the columns of A. This way we get linear combinations of the columns of A:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 3 \end{pmatrix} + y \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

Example 11.1. *What is the matrix associated to the linear function*

$$f(x_1, x_2, x_3, x_4) = (x_1 + 2x_2 + 3x_3 - x_4, -x_1 - 3x_2 - x_3, 3x_1 + 4x_2 + 14x_3)?$$

This functions takes as input a vector with 4 entries and spits out a vector with 3 entries,

$$f \colon \mathbb{R}^4 \longrightarrow \mathbb{R}^3.$$

So we are looking for a matrix A with shape 3×4. We just need to calculate the images of the four vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ and $\vec{e}_4$:

$$\begin{aligned} f(1,0,0,0) &= (1,-1,3) \\ f(0,1,0,0) &= (2,-3,4) \\ f(0,0,1,0) &= (3,-1,14) \\ f(0,0,0,1) &= (-1,0,0). \end{aligned}$$

So

$$A = \begin{pmatrix} 1 & 2 & 3 & -1 \\ -1 & -3 & -1 & 0 \\ 3 & 4 & 14 & 0 \end{pmatrix}.$$

Example 11.2. *Is the function f one-to-one?*

We are asking if the matrix equation

$$A\vec{x} = \vec{b}$$

always has at most one solution. This is the same as to ask if the matrix equation:

$$A\vec{x} = \vec{0}$$

has only the obvious (aka trivial) solution $(x_1, x_2, x_3, x_4) = (0, 0, 0, 0)$.

We apply Gaussian elimination. We multiply the first row by 1 and -3 and add it to the second and third rows:

$$\begin{pmatrix} 1 & 2 & 3 & -1 \\ -1 & -3 & -1 & 0 \\ 3 & 4 & 14 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -2 & 5 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & -2 & 5 & 3 \end{pmatrix}$$

To get from the second matrix to the third matrix we multiplied by the second row by -1. Now we multiply the second row by 2 and add it to the third row:

$$\begin{pmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & -2 & 5 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 5 \end{pmatrix}.$$

Now imagine solving these equations by back substitution. x_1, x_2 and x_3 are basic variables, so that what is left, x_4 is a free variable. But this means that the homogeneous equation $A\vec{x} = \vec{0}$ has a non-obvious solution.

If we carry out back substitution we get:

$$x_3 + 5x_4 = 0 \qquad \text{so that} \qquad x_3 = -5x_4.$$

Using this we get:

$$x_2 + 10x_4 + x_4 = 0 \qquad \text{so that} \qquad x_2 = -11x_4.$$

Finally we get:

$$x_1 - 22x_4 - 15x_4 - x_4 = 0 \qquad \text{so that} \qquad x_2 = 38x_4.$$

So the general solution to the homogeneous is:

$$(x_1, x_2, x_3, x_4) = (38x_4, -11x_4, -5x_4, x_4) = x_4(38, -11, -5, 1),$$

a line through the origin in $\mathbb{R}^4$.

Example 11.3. *Are the vectors* $(1, -1, 3)$, $(2, -3, 4)$, $(3, -1, -14)$ *and* $(-1, 0, 0)$ *independent?*

We are asking if we can find x_1, x_2, x_3, x_4, not all zero, so that

$$x_1 \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ -1 \\ -14 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Let A be the matrix whose columns are the vectors $(1, -1, 3)$, $(2, -3, 4)$, $(3, -1, -14)$ and $(-1, 0, 0)$. We are asking if we can find a non-obvious solution to

$$A\vec{x} = \vec{0}.$$

We have already seen we can. The question whether the vectors are independent is the same as whether the function is one-to-one.

Example 11.4. *Do the vectors* $(1, -1, 3)$, $(2, -3, 4)$, $(3, -1, -14)$ *and* $(-1, 0, 0)$ *span* $\mathbb{R}^3$*?*

Given a vector $\vec{b} \in \mathbb{R}^3$ we are asking if we can find x_1, x_2, x_3 and x_4 such that

$$x_1 \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ -1 \\ -14 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

Equivalently can we always solve

$$A\vec{x} = \vec{b}?$$

Is this equation always consistent?

Apply Gaussian elimination. We never get a pivot in the last column of the augmented matrix. So the equation is always consistent.

Example 11.5. *Is* $(x_1, x_2, x_3, x_4) = (1, 1, 1, 1)$ *a solution to*

$$\begin{aligned} x_1 + 2x_2 + 3x_3 - x_4 &= 5 \\ -x_1 - 3x_2 - x_3 \qquad &= -5 \\ 3x_1 + 4x_2 + 14x_3 \qquad &= 21? \end{aligned}$$

What is the general solution?

We check:

$$\begin{aligned} 1 + 2 + 3 - 1 &= 5 \\ -1 - 3 - 1 \qquad &= -5 \\ 3 + 4 + 14 \qquad &= 21. \end{aligned}$$

So $(1, 1, 1, 1)$ a solution.

The general solution is a particular solution plus the general solution to the homogeneous.

$$(x_1, x_2, x_3, x_4) = (1, 1, 1, 1) + x_4(38, -11, -5, 1).$$

This is a translation of a line through the origin.

Example 11.6. *What is the inverse of the matrix*

$$A = \begin{pmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{pmatrix}?$$

We apply Gauss-Jordan elimination. We multiply the first row by 3 and -2 and add it to the second and third rows.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right)$$

To get from the second matrix to the third we multiplied the second row by 3 and added it to the third row. Next we divide the third row by 2:

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array}\right)$$

For the last step, we multiplied the third row by 2 and added it to the second and first rows.

To check that

$$C = \begin{pmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{pmatrix}$$

let's compute a few entries of the product AC and CA:

If we take the first row of A and multiply by the first column of C we get

$$1 \cdot 8 + 0 \cdot 10 + (-2) \cdot 7/2 = 1,$$

as expected.

If take the second row of C and multiply it by the third column of A we get

$$10 \cdot -2 + 4 \cdot 4 + 1 \cdot 4 = 0,$$

as expected.