15. Coordinate systems

If we choose a basis then every vector can uniquely written as a sum of the basis vectors.

For example, suppose we consider the basis $\vec{v}_1 = (1, 1)$ and $\vec{v}_2 = (1, -1)$ of $\mathbb{R}^2$. Consider the vector $\vec{b} = (3, 1)$. Let's try to write this as a linear combination of $\vec{v}_1$ and $\vec{v}_2$:

$$\vec{b} = c_1\vec{v}_1 + \vec{v}_2.$$

As usual this gives us a system of linear equations:

$$P\vec{c} = \vec{b},$$

where $\vec{c} = (c_1, c_2)$, P is the matrix whose columns are $\vec{v}_1$ and $\vec{v}_2$ and $\vec{b} = (3, 1)$. P is called the change of coordinate matrix. We solve the augmented matrix using Gaussian elimination:

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & 1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -2 & -2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 1 \end{array}\right).$$

So $c_2 = 1$ and

$$c_1 + 1 = 3 \qquad \text{so that} \qquad c_1 = 2.$$

Note that this makes sense geometrically.

Using these ideas, we can figure out the dimension of P_n, the vector space of polynomials of degree n. Every polynomial of degree at most n can be uniquely written as a sum

$$a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n.$$

It follows that the polynomials 1, t, t^2, ..., t^n are a basis of P_n. P_n has dimension $n + 1$. In coordinates, the polynomial

$$a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n,$$

corresponds to the vector

$$(a_0, a_1, \dots, a_n) \in \mathbb{R}^{n+1}.$$