24. This and that

Recall the Fibonacci sequence

$$f_0 = 0, 1, 1, 2, 3, 5, 8, 13, \cdots, f_n, \cdots,$$

satisfies the recurrence

$$f_n = f_{n-2} + f_{n-1}.$$

If

$$\vec{v}_n = \begin{pmatrix} f_{n-1} \\ f_n \end{pmatrix}.$$

then

$$\vec{v}_{n+1} = \begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} = \begin{pmatrix} f_n \\ f_{n-1} + f_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} f_{n-1} \\ f_n \end{pmatrix}.$$

Thus if

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \vec{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

then $\vec{v}_n = A^{n-1}\vec{v}_1$.

Let's diagonalise A:

$$A - \lambda I_2 = \begin{pmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix}.$$

The characteristic equation is

$$-\lambda(1-\lambda) - 1 = 0.$$

The quadractic polynomial on the LHS is the characteristic polynomial. Expanding, we get

$$\lambda^2 - \lambda - 1 = 0.$$

Using the quadratic formula gives

$$\lambda = \frac{1 \pm \sqrt{5}}{2}.$$

Note that the golden ratio:

$$\frac{1+\sqrt{5}}{2}.$$

turns up as one of the roots. If we plug in $\lambda_1 = (1+\sqrt{5})/2$ then we get

$$\begin{pmatrix} -\frac{1+\sqrt{5}}{2} & 1 \\ 1 & \frac{1-\sqrt{5}}{2} \end{pmatrix}.$$

Let's apply Gaussian elimination:

$$\begin{pmatrix} -\frac{1+\sqrt{5}}{2} & 1 \\ 1 & \frac{1-\sqrt{5}}{2} \end{pmatrix} \to \begin{pmatrix} 1 & \frac{1-\sqrt{5}}{2} \\ 1 & \frac{1-\sqrt{5}}{2} \end{pmatrix} \to \begin{pmatrix} 1 & \frac{1-\sqrt{5}}{2} \\ 0 & 0 \end{pmatrix},$$

so that this is indeed a matrix of rank one. The kernel is spanned by

$$\vec{v}_1 = (1, \frac{1+\sqrt{5}}{2}).$$

This is an eigenvector with eigenvalue λ_1. Similarly

$$\vec{v}_2 = (1, \frac{1-\sqrt{5}}{2}).$$

is an eigenvector with eigenvalue

$$\lambda_2 = \frac{1-\sqrt{5}}{2}.$$

Thus $A = PDP^{-1}$, where

$$D = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}$$

and

$$P = \begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix}.$$

It follows that

$$P^{-1} = -\frac{1}{\sqrt{5}} \begin{pmatrix} \frac{1-\sqrt{5}}{2} & -1 \\ -\frac{1+\sqrt{5}}{2} & 1 \end{pmatrix}$$

One can check the equality $A = PDP^{-1}$. Now

$$\begin{aligned}
A^n v_1 &= PD^nP^{-1}v_1 \\
&= \frac{1}{-\sqrt{5}} \begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{pmatrix} \begin{pmatrix} \frac{1-\sqrt{5}}{2} & -1 \\ -\frac{1+\sqrt{5}}{2} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
&= \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n \\ -\left(\frac{1-\sqrt{5}}{2}\right)^n \end{pmatrix} \\
&= \frac{1}{\sqrt{5}} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \\ ? \end{pmatrix}.
\end{aligned}$$

It follows that

$$f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Now

$$-1 < \frac{1-\sqrt{5}}{2} < 0 \qquad \text{whilst} \qquad \frac{1+\sqrt{5}}{2} > 1.$$

If n is large this means

$$\left(\frac{1-\sqrt{5}}{2}\right)^n \approx 0.$$

and the other term is the one that matters. But f_n is an integer. It follows that f_n is the closest integer to

$$\left(\frac{1}{\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

It is interesting to check this for some values of n. Put in $n = 5$ and we get

$$\approx 4.956,$$

which is very close to the real answer, namely 5. Put in $n = 6$ and we get

$$\approx 8.025,$$

which is even closer to the real answer, namely 8. Put in $n = 100$ (well into matlab, or your favourite computer algebra system) we get

$$3.542248 \times 10^{20}.$$

Actually this is nowhere near the real answer. Matlab (or YFCAS) has a function to compute f_{100} directly (and more importantly correctly).

Here is what is going on. To compute f_{100} accurately using matrices, which involves real numbers, we need twenty significant figures of accuracy. Matlab, let's say, routinely uses ten significant figures of accuracy, so only the first ten digits are correct.

On the other hand, the routine which matlab uses to compute the Fibonacci numbers, does the stupid thing and just keeps computing each term in the sequence until it gets to a hundred. The advantage of this is that the computer knows exactly how much accuracy it needs as it computes; if it has an integer like 1450 it needs four significant figures but if it has a number like 123456 it needs six, and so on.