# 25. Orthogonal projection

What is the distance between a point $p$ and a plane $H$ in $\mathbb{R}^3$? What is the distance between a point $p$ and a line $L$ in $\mathbb{R}^3$? In the first case we want a point $q \in H$ such that the line $pq$ is orthogonal to $H$. Similarly we want a point $q$ on $L$ such that the line $pq$ is orthogonal to $L$.

Let's use vectors to solve this problem. Let's assume that $H$ and $l$ contain the origin, so that they are linear subspaces. We first treat the second problem of a line through the origin. In this case the line $L$ is the span of a single vector $\vec{u}$.

The point $p$ is represented by a vector $\vec{y}$. The orthogonal projection $q$ is a point of the line $L$ so that there is a scalar $\alpha$ such that the vector corresponding to $q$ is $\vec{y}_0 = \alpha\vec{u}$. What is left over,

$$\vec{y}_1 = \vec{y} - \alpha\vec{u}$$

is orthogonal to $\vec{u}$. So

$$0 = \vec{y}_1 \cdot \vec{u} = (\vec{y} - \alpha\vec{u}) \cdot \vec{u} = \vec{y} \cdot \vec{u} - \alpha\vec{u} \cdot \vec{u}.$$

Thus

$$\alpha = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}.$$

The orthogonal projection of $\vec{y}$ onto $L$ is then the vector

$$\vec{y}_0 = \operatorname{proj}_L \vec{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u}.$$

Note that this formula is valid in $\mathbb{R}^n$.

**Example 25.1.** *What is the distance between the point $p = (1, -1, 2)$ and the line $L$ given parametrically as $(t, 2t, 3t)$?*

Let $\vec{y} = (1, -1, 2)$. We want the point $q$ on the line $L$ closest to $p$. The corresponding vector $\vec{y}_0$ is a multiple of $\vec{u} = (1, 2, 3)$.

$$\vec{y}_0 = \operatorname{proj}_L \vec{y} = \frac{(1, -1, 2) \cdot (1, 2, 3)}{(1, 2, 3) \cdot (1, 2, 3)}(1, 2, 3) = \frac{5}{14}(1, 2, 3).$$

We want the distance between $(1, -1, 2)$ and $\frac{5}{14}(1, 2, 3)$:

$$\sqrt{\frac{81}{14^2} + \frac{144}{49} + \frac{169}{14^2}} = \sqrt{\frac{125}{2 \cdot 49} + \frac{144}{49}} = \sqrt{\frac{59}{14}}.$$

Now let's turn to the first problem, the distance between a point $p$ and a plane $H$. Suppose that $H$ is the span of two orthogonal vectors $\vec{u}_1$ and $\vec{u}_2$, so that

$$\{\, \vec{u}_1, \vec{u}_2 \,\}$$

is an orthogonal basis of $H$.

As before let $\vec{y}$ be the vector corresponding to $p$ and let $\vec{y}_0 \in H$ be the closest vector to $\vec{y}$. Then $\vec{y}_1 = \vec{y} - \vec{y}_0$ is orthogonal to $H$, so that it is orthogonal to $\vec{u}_1$ and $\vec{u}_2$.

As $\vec{y}_0 \in H$ and $\{\, \vec{u}_1, \vec{u}_2 \,\}$ is a basis of $H$ we may find scalars $\alpha_1$ and $\alpha_2$ such that

$$\vec{y}_0 = \operatorname{proj}_H \vec{y} = \alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2.$$

We know that

$$\vec{y}_1 = \vec{y} - \vec{y}_0 = \vec{y} - (\alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2)$$

is orthogonal to $H$, that is, it is orthogonal to $\vec{u}_1$ and $\vec{u}_2$. Thus

$$0 = \vec{y}_1 \cdot \vec{u}_1 = (\vec{y} - \alpha_1 \vec{u}_1 - \alpha_2 \vec{u}_2) \cdot \vec{u}_1 = \vec{y} \cdot \vec{u}_1 - \alpha_1 \vec{u}_1 \cdot \vec{u}_1.$$

Solving for $\alpha_1$ we get:

$$\alpha_1 = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}.$$

The same piece of algebra with the subscript $_2$ replacing the subscript $_1$ yields:

$$\alpha_2 = \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2}.$$

**Example 25.2.** *What is the distance between the point $p = (3, 1, 5, 1)$ and the plane spanned by the vectors $\vec{u}_1 = (3, 1, -1, 1)$ and $\vec{u}_2 = (1, -1, 1, -1)$ in $\mathbb{R}^4$?*

We check that $\vec{u}_1$ and $\vec{u}_2$ are orthogonal:

$$\vec{u}_1 \cdot \vec{u}_2 = (3, 1, -1, 1) \cdot (1, -1, 1, -1) = 0.$$

So $\vec{u}_1$ and $\vec{u}_2$ are orthogonal. We try to write $\vec{y} = (3, 1, 5, 1)$ as a sum $\vec{y}_0 + \vec{y}_1$, where

$$\vec{y}_0 = \operatorname{proj}_H \vec{y} = \alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2.$$

We have

$$\alpha_1 = \frac{(3, 1, 5, 1) \cdot (3, 1, -1, 1)}{(3, 1, -1, 1) \cdot (3, 1, -1, 1)} = \frac{6}{12} = \frac{1}{2}$$

and

$$\alpha_2 = \frac{(3, 1, 5, 1) \cdot (1, -1, 1, -1)}{(1, -1, 1, -1) \cdot (1, -1, 1, -1)} = \frac{6}{4} = \frac{3}{2}.$$

Thus the closest vector to $\vec{y} = (3, 1, 5, 1)$ in $H$ is

$$\vec{y}_0 = \operatorname{proj}_H \vec{y} = \frac{1}{2}(3, 1, -1, 1) + \frac{3}{2}(1, -1, 1, -1) = (3, -1, 1, -1).$$

The distance between $p$ and $H$ is the length of the vector

$$\vec{y}_1 = \vec{y} - \vec{y}_0 = (3, 1, 5, 1) - (3, -1, 1, -1) = (0, 2, 4, 2) = 2(0, 1, 2, 1).$$

The distance is

$$2\sqrt{1 + 4 + 1} = 2\sqrt{6}.$$

There are general results along these lines whose proofs are simple generalisations of the arguments above:

**Definition-Theorem 25.3.** *If $W$ is a subspace of $\mathbb{R}^n$ and $\vec{y}$ is a vector in $\mathbb{R}^n$ then we may uniquely decompose*

$$\vec{y} = \vec{y}_0 + \vec{y}_1$$

*where $\vec{y}_0 \in W$ and $\vec{y}_1$ is orthogonal to $W$. $\vec{y}_0 = \operatorname{proj}_W \vec{y}$ is called the orthogonal projection of $\vec{y}$ onto $W$.*

*If*

$$\{\, \vec{u}_1, \vec{u}_2, \dots, \vec{u}_k \,\}$$

*is an orthogonal basis of $W$ then*

$$\vec{y}_0 = \operatorname{proj}_W \vec{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 + \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2}\vec{u}_2 + \cdots + \frac{\vec{y} \cdot \vec{u}_n}{\vec{u}_n \cdot \vec{u}_n}\vec{u}_n.$$

**Theorem 25.4.** *If $W$ is a subspace of $\mathbb{R}^n$ and $\vec{y}$ is a vector in $\mathbb{R}^n$ then the orthogonal projection $\vec{y}_0 = \operatorname{proj}_W \vec{y}$ is the closest point in $W$ to $\vec{y}$, that is,*

$$\|\vec{y} - \vec{y}_0\| \leq \|\vec{y} - \vec{w}\| \qquad \textit{for all} \qquad \vec{w} \in W,$$

*with equality if and only if $\vec{w} = \vec{y}_0$.*