29. Final review I

Example 29.1. *Find a basis of*

$$W = \{\, (a{+}2b{+}4c{+}8d, 2a{+}4b{+}6c{+}8d, 3a{+}6b{+}9c{+}12d) \mid (a,b,c,d) \in \mathbb{R}^4 \,\}.$$

Implicit in this question is the assumption that W is a linear subspace of $\mathbb{R}^3$. Given a matrix A there are three linear spaces one can attach to A, the column space, the row space and the null space.

There are many ways to see how to attach a matrix to W. One way is to realise that W is the image of the linear map

$$f \colon \mathbb{R}^4 \longrightarrow \mathbb{R}^3$$

given by

$$(a,b,c,d) \longrightarrow (a+2b+4c+8d, 2a+4b+6c+8d, 3a+6b+9c+12d).$$

There is a matrix A associate to f. It is the matrix A such that $f(\vec{x}) = A\vec{x}$. A is a 3×4 matrix. The columns of A correspond to the images of the four standard basis vectors.

$$\vec{e}_1 = (1,0,0,0), \qquad \vec{e}_2 = (0,1,0,0), \qquad \vec{e}_3 = (0,0,1,0) \qquad \text{and} \qquad \vec{e}_4 = (0,0,0,1).$$

The images are

$$f(1,0,0,0) = (1,2,3), \quad f(0,1,0,0) = (2,4,6), \quad f(\vec{e}_3) = (4,6,9) \quad \text{and} \quad f(\vec{e}_4) = (8,8,12).$$

Therefore

$$A = \begin{pmatrix} 1 & 2 & 4 & 8 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{pmatrix}.$$

Note that the row space and the null space of A are linear subspaces of $\mathbb{R}^4$. The column space of A is a linear subspace of $\mathbb{R}^3$. In fact the column space is indeed the image of the linear map f.

Here is another way to see all of this:

$$\begin{aligned}
\begin{pmatrix} a+2b+4c+8d \\ 2a+4b+6c+8d \\ 3a+6b+9c+12d \end{pmatrix} &= \begin{pmatrix} a \\ 2a \\ 3a \end{pmatrix} + \begin{pmatrix} 2b \\ 4b \\ 6b \end{pmatrix} + \begin{pmatrix} 4c \\ 6c \\ 9c \end{pmatrix} + \begin{pmatrix} 8d \\ 8d \\ 12d \end{pmatrix} \\
&= a \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + b \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + c \begin{pmatrix} 4 \\ 6 \\ 9 \end{pmatrix} + d \begin{pmatrix} 8 \\ 8 \\ 12 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 & 4 & 8 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.
\end{aligned}$$

Visibily then W is the column space of A. Therefore W is a linear subspace and it does make sense to look for a basis of W.

We apply Gaussian elimination:

$$\begin{pmatrix} 1 & 2 & 4 & 8 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 4 & 8 \\ 0 & 0 & -2 & -8 \\ 0 & 0 & -3 & -12 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 4 & 8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & -3 & -12 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 4 & 8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

There are pivots in the first and third rows. Gaussian elimination does not preserve the column space but it does preserve the relations. The first and third columns are a basis at the end of the elimination so the first and third columns of A are a basis of the column space.

The column space is the span of the vectors $(1, 2, 3)$ and $(4, 6, 9)$.

Note that $(1, 2, 4, 8)$ and $(0, 0, 1, 4)$ is basis for the row space. Gaussian elimination preserves the row space.

Example 29.2. *Given that* $(a, b, c, d) = (1, 1, 1, 1)$ *is a solution to the linear equations*

$$\begin{aligned} a + 2b + 4c + 8d &= 15 \\ 2a + 4b + 6c + 8d &= 20 \\ 3a + 6b + 9c + 12d &= 33, \end{aligned}$$

what is the general solution?

The general solution of the linear equations $A\vec{x} = \vec{b}$ is the sum of a particular solution plus the general solution of the homogeneous equation $A\vec{x} = \vec{0}$.

We know that A is row equivalent to

$$\begin{pmatrix} 1 & 2 & 4 & 8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We can find the general solution to the homogeneous by back substitution. a and c are basic variables, b and d are free variables. Hence

$$c + 4d = 0 \qquad \text{so that} \qquad c = -4d$$

and

$$a + 2b - 16d + 8d = 0 \qquad \text{so that} \qquad a = -2b + 8d.$$

The general solution of the homogeneous is

$$(a, b, c, d) = (-2b + 8d, b, -4d, d) = b(-2, 1, 0, 0) + d(8, 0, -4, 1).$$

The null space has a basis $(-2, 1, 0)$ and $(8, -4, 1)$.

The general solution to the linear equations in (29.2) is

$$(a, b, c, d) = (1, 1, 1, 1) + b(-2, 1, 0) + d(8, -4, 1).$$

Example 29.3. *If A is a 3×3 matrix and $\det A = 3$ what is $\det 2A$?*

The answer is

$$2^3 \cdot 3 = 24.$$

There are two ways to see this. The first is algebraic. If you multiply a row by 2 then the determinant gets multiplied by 2. A 3×3 matrix has three rows, so that each row is multiplied by 2. Thus the determinant gets multiplied by $2^3 = 8$.

The second is geometric. The absolute value of the determinant of A is the volume of a parallelepiped with sides given by the rows of A. If we multiply A by 2 then each side is doubled. Thus the volume goes up by a factor of 8.