3. Vectors and Matrices

We have already seen that to carry out Gaussian elimination it makes sense to use matrices. Can we go further and make sense of linear equations using matrix equations?

Given a system of linear equations

$$\begin{aligned} x + y + z &= 1 \\ 2x + y + z &= 2 \\ -3x - 2y + z &= 0, \end{aligned}$$

we have already seen that we should write down the coefficient matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -3 & -2 & 1 \end{pmatrix}$$

and the column vector

$$\vec{b} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

It also makes sense to make a column vector of the variables

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

(We will use arrows on top of vectors whenever we want to emphasize that they are vectors). Finding the solutions to the original system of linear equations is the same as finding all possible values of the vector $\vec{x}$. In terms of notation it makes sense to write down the matrix equation

$$A\vec{x} = \vec{b}.$$

We define the matrix product of A with $\vec{x}$ so that this comes out correctly. The rows of A correspond to the coefficients. So we take a row of A and multiply it by the column vector $\vec{x}$. We pair the entries of a row of A with the entries of a column of $\vec{b}$ and add them together to get a number.

In lecture 1 we checked that $(x, y, z) = (1, -1, 1)$ is a solution to the system of linear equations. Let's check this using matrix multiplication:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -3 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 1 \cdot -1 + 1 \cdot 1 \\ 2 \cdot 1 + 1 \cdot -1 + 1 \cdot 1 \\ -3 \cdot 1 - 2 \cdot -1 + 1 \cdot 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

Just to get some practice, let's check that $(x, y, z) = (3, 1, -2)$ is not a solution:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -3 & -2 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \cdot 3 + 1 \cdot 1 + 1 \cdot -2 \\ 2 \cdot 3 + 1 \cdot 1 + 1 \cdot -2 \\ -3 \cdot 3 - 2 \cdot 1 + 1 \cdot -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -13 \end{pmatrix},$$

nowhere close to a solution, as expected. Of course this vector is a solution to the equation

$$A\vec{x} = \vec{c} \qquad \text{where} \qquad \vec{c} = \begin{pmatrix} 2 \\ 5 \\ -13 \end{pmatrix}.$$

The interesting thing is that we can interpret matrix multiplication

$$A\vec{x}$$

in an entirely different way. Instead of focusing on the rows, lets focus on the columns:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -3 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \cdot x + 1 \cdot y + 1 \cdot z \\ 2 \cdot x + 1 \cdot y + 1 \cdot z \\ -3 \cdot x - 2 \cdot y + 1 \cdot z \end{pmatrix} = x \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + y \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + z \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For example, in terms of columns

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -3 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} - 1 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -3 & -2 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -13 \end{pmatrix},$$

as before.

Definition 3.1. *The vector $\vec{b} \in \mathbb{R}^m$ is a linear combination of the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \in \mathbb{R}^m$ if there are scalars $x_1, x_2, \dots, x_n$ such that*

$$\vec{b} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n.$$

The vectors

$$\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 2 \\ 5 \\ -13 \end{pmatrix}$$

are a linear combination of

$$\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Putting all of this together we see that

Theorem 3.2. *The matrix equation*

$$A\vec{x} = \vec{b}$$

is consistent if and only if $\vec{b}$ is a linear combination of the columns of A.

Example 3.3. *Is $(1, 1, -3)$ a linear combination of $(1, 0, 1)$, $(-1, 1, 1)$, $(1, 1, 1)$ and $(1, 2, 3)$?*

Let

$$A = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 3 \end{pmatrix}$$

be the matrix whose columns are the vectors $(1, 0, 1)$, $(-1, 1, 1)$, $(1, 1, 1)$ and $(1, 2, 3)$. Then we want to know if the matrix equation

$$A\vec{x} = \vec{b}$$

is consistent, where

$$\vec{b} = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} \qquad \text{and} \qquad \vec{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}.$$

To answer this question, write down the augmented matrix

$$B = \left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 3 & -3 \end{array}\right)$$

and apply Gaussian elimination. We eliminate the entries in the first column:

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 3 & -3 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 2 & 0 & 2 & -4 \end{array}\right).$$

Now the entries in the second column:

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 2 & 0 & 2 & -4 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & -2 & -2 & -6 \end{array}\right).$$

Finally we turn the -2 into a pivot:

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right).$$

Note that we can always solve this equation, since there are no pivots in the last column.

This answers the original question but let's also figure out how to write $(1, 1, -3)$ as a linear combination of $(1, 0, 1)$, $(-1, 1, 1)$, $(1, 1, 1)$ and $(1, 2, 3)$.

We apply back substitution. Note that w is a free variable. So we might as well set $w = 0$. In this case $z = 3$. But then

$$y + 3 = 1 \qquad \text{so that} \qquad y = -2.$$

Finally this says

$$x + 2 + 3 = 1 \qquad \text{so that} \qquad x = -4.$$

Thus

$$-4(1, 0, 1) - 2(-1, 1, 1) + 3(1, 1, 1) + 0(1, 2, 3) = (1, 1, -3).$$

Note that in fact any vector in $\mathbb{R}^3$ is a linear combination of $(1, 0, 1)$, $(-1, 1, 1)$, $(1, 1, 1)$ and $(1, 2, 3)$. Indeed, if we follow the same recipe as before then the coefficient matrix is unchanged, and we will get down to the same four columns of the echelon matrix (the fifth column will depend very much on the original vector $\vec{b}$ but this won't matter). As there are three pivots in the first three columns, there are no pivots in the fifth column and the linear system is consistent. In fact, by the same argument, every vector in $\mathbb{R}^3$ is a linear combination of the first three vectors, $(1, 0, 1)$, $(-1, 1, 1)$ and $(1, 1, 1)$.

Definition 3.4. *We say that the vectors $v_1, v_2, \dots, v_n \in \mathbb{R}^m$ spans $\mathbb{R}^m$ if every vector $\vec{b} \in \mathbb{R}^m$ is a linear combination of $v_1, v_2, \dots, v_n$.*

The vectors $(1, 0, 1)$, $(-1, 1, 1)$ and $(1, 1, 1)$ span $\mathbb{R}^3$.