4. Geometry of the span; homogeneous linear equations

It is interesting to think about the geometric meaning of linear combinations and the span. If you want to add two vectors $\vec{u}$ and $\vec{v}$, move the starting point of $\vec{v}$ to the endpoint of $\vec{u}$; the sum is the arrow you get by first going along $\vec{u}$ and then along $\vec{v}$.

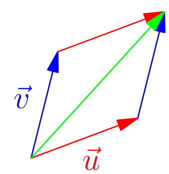

Figure 1. Addition of vectors: parallelogram rule

If λ is a scalar and $\vec{v}$ is a vector then $\lambda\vec{v}$ is the vector which is λ times as long as $\vec{v}$. If $\lambda > 0$ then $\lambda\vec{v}$ has the same direction as $\vec{v}$ and if $\lambda < 0$ then $\lambda\vec{v}$ has the opposite direction. Either way, we will say that $\lambda\vec{v}$ is parallel to $\vec{v}$.

The span of a collection of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ is just the set of all linear combinations of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$.

If we have one vector $\vec{v}$ the span consists of all multiples $\lambda\vec{v}$. The span of one non-zero vector $\vec{v}$ is just the set of vectors parallel to this vector (the span of the zero vector is just the origin). This is the same thing as a line through the origin.

For example the span of the vector $(1, 0, 0)$ consists of all vectors $(\lambda, 0, 0)$, the span is the x-axis, the span of the vector $(0, 1, 0)$ is the y-axis and the span of the vector $(0, 0, 1)$ is the z-axis. The span of the vector $(1, 1, 1)$ is the set of all vectors $(\lambda, \lambda, \lambda)$; it is the line through the origin going through the point $(1, 1, 1)$.

If $\vec{v}$ and $\vec{w}$ are two vectors then the span of these vectors consists of all linear combinations

$$\lambda\vec{v} + \mu\vec{w}.$$

Now $\lambda\vec{v}$ is parallel to $\vec{v}$ and $\mu\vec{w}$ is parallel to $\vec{w}$. If $\vec{v}$ and $\vec{w}$ are non-zero then we get two lines through the origin. The sum gives a vector in the plane containing these two lines. The span of two vectors is nearly always a plane.

For example the span of the vectors $(1, 0, 0)$ and $(0, 1, 0)$ consists of all vectors $(\lambda, \mu, 0)$–it is the xy-plane (the floor), the plane $z = 0$; the span of the vectors $(1, 0, 0)$ and $(0, 0, 1)$ consists of all vectors $(\lambda, 0, \mu)$–it is the xz-plane (a vertical coordinate plane), the plane $y = 0$; the span of the vectors $(0,1,0)$ and $(0,0,1)$ consists of all vectors $(0,\lambda,\mu)$–it is the yz-plane (the other vertical coordinate plane), the plane $x = 0$. The vectors $(1,-1,0)$ and $(1,0,-1)$ span a plane–it is the plane $x+y+z = 0$.

What is the span of the vectors $(1,0,0)$ and $(2,0,0)$? Both vectors span the same line, the x-axis and so their span is the x-axis. The problem is that the two vectors are parallel. In general two vectors span a plane, unless the two vectors are parallel, when you just get a line.

If we have three equations in three unknowns, how many different configurations of planes do we get? One possibility is the three planes meet at a single point; something like the three planes $x = 0$, $y = 0$ and $z = 0$. But what if two of the planes are parallel? One plane could be $z = 0$ and the other could be $z = 1$. Perhaps the third plane is $x = 0$. The plane meets the other two planes in two parallel line.

To count, we have to be more systematic. Let's group the configurations by the solution set. We could get no solutions, a point, a line and a plane. There is only way to get a plane, the three planes we started with are the same, $x = 0$, $x = 0$ and $x = 0$. There is also only one way to get a point. How do we get a line? All three planes must contain the line. One possibility is to have three different planes containing the line. The planes $x = 0$, $y = 0$ and $x + y = 0$ all contain the z-axis. But we could have the same plane twice, plus another plane, $x = 0$, $x = 0$ and $y = 0$ (the intersection is the z-axis). Finally how do we get no solutions? One possibility is two have two parallel planes and a third plane. We could also have three parallel planes, $z = 0$, $z = 1$ and $z = 2$. One of the parallel planes could be repeated $z = 0$, $z = 0$ and $z = 1$. There is one final possibility. We could have three planes which intersect in three parallel lines (the "toblerone solution"). The planes $x = y$, $y = 0$ and $x + y = 2$ (so that planes of the form $z = a$ give the same cross-section; three lines which give triangle).

Adding the four possibilities together, no solutions, a point, a line and a plane there are

$$4 + 1 + 2 + 1 = 8$$

different configurations.

A matrix equation of the form

$$A\vec{x} = \vec{0}$$

is called homogeneous. Here $\vec{0}$ is the column vector with m rows, all zero. The zero vector $\vec{x} = \vec{0}$ (now with n rows) is always a solution to a homogeneous equation,

$$A\vec{0} = \vec{0}.$$

Thus a homogeneous equation is always consistent. The solution $\vec{x} = \vec{0}$ is sometimes called the trivial solution. There are infinitely many solutions if and only if there are free variables.

Example 4.1. *Does the system*

$$\begin{aligned} x + 2y + 3z &= 0 \\ 2x + 3y + 4z &= 0 \\ 4x + 7y + 10z &= 0, \end{aligned}$$

have a non-trivial solution?

We just have to see if there are any free variables. We write down the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 0 \\ 4 & 7 & 10 & 0 \end{array}\right).$$

Now we apply Gaussian elimination:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 0 \\ 4 & 7 & 10 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

z is a free variable. There are infinitely many solutions. In particular there are non-trivial solutions.

If we solve by back substitution we get

$$y + 2z = 0 \qquad \text{so that} \qquad y = -2z,$$

and

$$x - 4z + 3z = 0 \qquad \text{so that} \qquad x = z$$

The general solution is

$$(x, y, z) = (z, -2z, z) = z(1, -2, 1),$$

a line through the origin, the span of the vector $(1, -2, 1)$.

Now suppose that we keep the same coefficient matrix but change $\vec{b}$. Let's solve

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x + 3y + 4z &= 9 \\ 4x + 7y + 10z &= 21. \end{aligned}$$

We write down the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & 3 & 4 & 9 \\ 4 & 7 & 10 & 21 \end{array}\right)$$

Now we apply Gaussian elimination:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & 3 & 4 & 9 \\ 4 & 7 & 10 & 21 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -1 & -2 & -3 \\ 0 & -1 & -2 & -3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & -2 & -3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The system is consistent. z is a free variable.

If we solve by back substitution we get

$$y + 2z = 3 \qquad \text{so that} \qquad y = 3 - 2z,$$

and

$$x + 6 - 4z + 3z = 6 \qquad \text{so that} \qquad x = z.$$

The general solution is

$$(x, y, z) = (z, 3 - 2z, z) = (0, 3, 0) + z(1, -2, 1),$$

which represents a line. Notice that this line is a translate of the line $z(1, -2, 1)$. The two lines

$$(0, 3, 0) + z(1, -2, 1) \qquad \text{and} \qquad z(1, -2, 1)$$

are parallel.

This is a general phenomena. The solutions to the matrix equation

$$A\vec{x} = \vec{b}$$

are of the form

$$\vec{x} = \vec{x}_h + \vec{x}_p.$$

Here $\vec{x}_p$ is a fixed vector, one solution to the original equation

$$A\vec{x} = \vec{b}.$$

$\vec{x}_h$ is then any solution to the homogeneous equation

$$A\vec{x} = \vec{0}.$$

In the example above,

$$\vec{x}_p = (0, -3, 0)$$

is a particular solution and

$$\vec{x}_h = z(1, 2, 1),$$

is the general solution to the homogeneous equation.

Consider the plane

$$x + y + z = 1.$$

Let's find the general solution. We apply Gaussian elimination

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \end{array}\right).$$

The elimination is complete. We solve by back substitution. y and z are free variables.

$$x + y + z = 1 \qquad \text{so that} \qquad x = 1 - y - z.$$

The general solution is

$$\begin{aligned}(x, y, z) &= (1 - y - z, y, z)\\ &= (-y, y, 0) + (-z, 0, z) + (1, 0, 0)\\ &= y(1, -1, 0) + z(1, 0, -1) + (1, 0, 0).\end{aligned}$$

Here

$$\vec{x}_p = (1, 0, 0),$$

is a particular solution.

$$\vec{x}_h = y(1, -1, 0) + z(1, 0, -1),$$

is a plane through the origin, the general solution to the homogeneous equation

$$x + y + z = 0.$$