5. Dependence and Independence

We saw that the homogeneous system of equations

$$\begin{aligned} x + 2y + 3z &= 0 \\ 2x + 3y + 4z &= 0 \\ 4x + 7y + 10z &= 0, \end{aligned}$$

has a non-trivial solution. That is, we saw that there are infinitely many solutions. But there are three equations and three unknowns, so we'd expect only one solution. What went wrong?

If you follow the steps of Gaussian elimination carefully, you will see that the third equation is twice the first equation plus the second equation. In terms of matrices, if we look at the rows of the coefficient matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 7 & 10 \end{pmatrix}$$

the third row is twice the first row plus the second row.

In this case, the vector $\vec{v}_3 = (4, 7, 10)$ is a linear combination of the vectors $\vec{v}_1 = (1, 2, 3)$ and $\vec{v}_2 = (2, 3, 4)$. The vector $\vec{v}_3 = (4, 7, 10)$ belongs to the span of the vectors $\vec{v}_1 = (1, 2, 3)$ and $\vec{v}_2 = (2, 3, 4)$. For obvious reasons we say that the vectors $\vec{v}_1 = (1, 2, 3)$, $\vec{v}_2 = (2, 3, 4)$ and $\vec{v}_3 = (4, 7, 10)$ are dependent.

Now mathematicians like symmetry and they like the number zero. So let's rewrite the dependence

$$\vec{v}_3 = 2\vec{v}_1 + \vec{v}_2$$

so that it is symmetric. We can do this using the vector zero,

$$2\vec{v}_1 + \vec{v}_2 - \vec{v}_3 = \vec{0} = (0, 0, 0).$$

Definition 5.1. *The vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \in \mathbb{R}^m$ are (linearly) dependent if there are scalars $x_1, x_2, \dots, x_m$, not all zero, such that*

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}.$$

We say that the vectors are (linearly) independent if they are not dependent.

As usual, all of this can be restated in terms of matrix equations:

Theorem 5.2. *The columns of the matrix A are dependent if and only if the homogeneous equation*

$$A\vec{x} = \vec{0}$$

has a non-trivial solution.

One can restate (5.2) in terms in terms of independence:

The columns of the matrix A are independent if and only if the only solution of the homogeneous equation

$$A\vec{x} = \vec{0}$$

is the trivial solution.

Example 5.3. *Are the vectors* $\vec{v}_1 = (1, 2, 3)$, $\vec{v}_2 = (2, 3, 4)$ *and* $\vec{v}_3 = (4, 7, 10)$ *independent?*

Let A be the matrix whose columns are $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$.

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ 3 & 4 & 10 \end{pmatrix}$$

We want to know if the homogeneous equation

$$A\vec{x} = \vec{0}$$

has a non-trivial solution. We apply Gaussian elimination

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 2 & 3 & 7 & 0 \\ 3 & 4 & 10 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Since there are only two pivots and three variables, there is a free variable (in fact z is a free variable). So the vectors are dependent.

Example 5.4. *Are the vectors* $\vec{v}_1 = (1, 0, 0)$, $\vec{v}_2 = (-20, 1, 0)$ *and* $\vec{v}_3 = (\pi, 2^e, 1)$ *independent?*

Let A be the matrix whose columns are $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$.

$$\begin{pmatrix} 1 & -20 & \pi \\ 0 & 1 & 2^e \\ 0 & 0 & 1 \end{pmatrix}.$$

We want to know if the homogeneous equation

$$A\vec{x} = \vec{0}$$

has a non-trivial solution. This equation does not have a non-trivial solution as there are three pivots and three variables. So there are no free variables, there is only one solution, the trivial solution $(0, 0, 0)$. The vectors are independent.

It is interesting to consider the geometric meaning of dependence. Suppose that we have one vector $\vec{v}$. We want to write $\vec{0}$ as a non-trivial multiple of $\vec{v}$. The only possibility is that $\vec{v} = \vec{0}$. One vector is linearly dependent if and only if it is the zero vector.

In fact we have the following silly result:

Lemma 5.5. *If one of* $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ *is the zero vector then* $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ *is linearly dependent.*

Proof. Suppose that the ith vector is the zero vector. Take $x_i = 1$ and every other scalar to be zero. Then not all of $x_1, x_2, \dots, x_n$ are zero and

$$\begin{aligned} x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n &= 0\vec{v}_1 + \cdots + 0\vec{v}_{i-1} + 1 \cdot \vec{0} + 0\vec{v}_{i+1} + \cdots + 0\vec{v}_n \\ &= \vec{0}. \end{aligned}$$

□

Now suppose that we have two non-zero vectors $\vec{v}$ and $\vec{w}$. If $\vec{v}$ and $\vec{w}$ are parallel then one is a multiple of the other:

$$\vec{w} = \lambda\vec{v}.$$

It is clear that they are dependent. In fact one can rearrange the equation above so that

$$-\lambda\vec{v} + \vec{w} = \vec{0}.$$

Now suppose that the two vectors $\vec{v}$ and $\vec{w}$ are dependent. Then we can find λ and μ scalars, not both zero, so that

$$\lambda\vec{v} + \mu\vec{w} = \vec{0}.$$

We can rewrite this as

$$\lambda\vec{v} = \mu\vec{w}.$$

Suppose that $\lambda \neq 0$. Then

$$\vec{v} = (\mu/\lambda)\vec{w}$$

so that $\vec{v}$ and $\vec{w}$ are parallel. Similarly, if $\mu \neq 0$ then $\vec{v}$ and $\vec{w}$ are parallel. Putting all of this together:

Proposition 5.6. *Two non-zero vectors* $\vec{v}$ *and* $\vec{w}$ *are dependent if and only if they are parallel.*

There is one other easy case when a collection of vectors is always dependent.

Theorem 5.7. *If* $m < n$ *then* n *vectors in* $\mathbb{R}^m$ *are always dependent.*

Proof. Let A be the matrix whose columns are the n vectors. We want to show that the homogeneous equation

$$A\vec{x} = \vec{0}$$

has a non-trivial solution. We apply Gaussian elimination. As there are m rows there are at most m pivots. As there are n variables some variables are free variables. But then the homogeneous equation has a non-trivial solution. □

Note that (5.6) makes geometric sense. If $v_1, v_2, \dots, v_n$ are independent then the first vector spans a line, the first two vectors span a plane, the first three vectors span a 3-plane, so that they all span an n-plane. But you cannot fit an n-plane into $\mathbb{R}^m$, just like you cannot fit a plane into a line, three space into a plane and so on.

There is one other key result:

Proposition 5.8. *The vectors $v_1, v_2, \dots, v_n$ are dependent if and only if one vector is a linear combination of the other vectors.*

Proof. Suppose that the vectors $v_1, v_2, \dots, v_n$ are dependent. Then we may find scalars $x_1, x_2, \dots, x_n$, not all zero, so that

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_n\vec{v}_n = \vec{0}.$$

Suppose that $x_i \neq 0$. Take $x_i\vec{v}_i$ over to the other side:

$$-x_i\vec{v}_i = x_1\vec{v}_1 + x_2\vec{\vec{v}}_2 + \dots + x_n\vec{v}_n.$$

If we divide through by $-1/x_i$ then we get

$$\vec{v}_i = (-x_1/x_i)\vec{v}_1 + (-x_2/x_i)\vec{v}_2 + \dots (-x_n/x_i)\vec{v}_n.$$

Thus $\vec{v}_i$ is a linear combination of the other vectors.

Now suppose that $\vec{v}_i$ is a linear combination of the other vectors:

$$\vec{v}_i = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n.$$

If we take $\vec{v}_i$ over to the other side we get

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n - \vec{v}_i$$

Note that the coefficient of $\vec{v}_i$ is -1, which isn't zero. So the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are dependent. ∎

Example 5.9. *Suppose that we consider*

$$\vec{v}_1 = (1,0) \qquad \vec{v}_2 = (1,0) \qquad \textit{and} \qquad \vec{v}_3 = (0,1).$$

Then the vectors are dependent.

$$\vec{v}_1 - \vec{v}_2 + 0 \cdot v_3 = \vec{0}.$$

We can write $\vec{v}_2$ as a linear combination of $\vec{v}_1$ and $\vec{v}_3$. Indeed, $\vec{v}_2 = \vec{v}_1$. But we cannot write $\vec{v}_3$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$.