9. Elementary matrices

Here we present a slightly different way to think about Gauss-Jordan elimination and inverses. Given any row operation, there is a matrix E which represents this row operation. This is best illustrated with examples. Let

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 4 & -5 \\ 2 & -3 & 5 \end{pmatrix}$$

Recall that there are three types of row operations:

- add a multiple of one row to another,
- multiply one row by a scalar,
- swap two rows.

Let's say we want to add the first row the second row. We multiply by

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then EA is

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ -1 & 4 & -5 \\ 2 & -3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -3 \\ 2 & -3 & 5 \end{pmatrix}$$

Now suppose we want to multiply the first row by -2 and add it to the third row: We multiply by

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -3 \\ 2 & -3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -3 \\ 0 & -1 & 1 \end{pmatrix}$$

Let's suppose we want to swap the second and third row. Let

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -3 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & -1 & 1 \\ 0 & 3 & -3 \end{pmatrix}$$

Finally if we want to multiply the second row by -1 we multiply by

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 0 & -1 & 1 \\ 0 & 3 & -3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 3 & -3 \end{pmatrix}.$$

The matrices corresponding to the elementary row operations are called elementary matrices. Given an operation the corresponding row operation is the effect of applying the row operation to the identity.

For example if we take I_3, multiply the first row by -2 and add it to the third row we get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}.$$

If we can write A as a product of elementary matrices $E_1 E_2 E_3 \dots E_k$ then the inverse of A is

$$E_k^{-1} E_{k-1}^{-1} \dots E_2^{-1} E_1^{-1}.$$

Note that the inverse of an elementary matrix is an elementary matrix. To undo multiplying the first row by -2 and adding it to the third row, multiply the first row by 2 and add it to the third row:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$