You have 50 minutes. This test is closed book, closed notes, no calculators.

There are 7 problems, and the total number of points is 100. Show all your work. Please make your work as clear and easy to follow as possible.

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Signature:______________________                                   
Student ID #:___________________                                   
Dissertation instructor:________________                            
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1. (20pts) (i) Show that the matrix 
\[
\begin{pmatrix}
1 & -1 & 2 & 3 \\
2 & -2 & 5 & 1 \\
-1 & 1 & -5 & 12
\end{pmatrix}
\] 
has echelon form 
\[
\begin{pmatrix}
1 & -1 & 2 & 3 \\
0 & 0 & 1 & -5 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

To get full credit for this problem, you must show your steps and explain what row operations you are using at each stage.

**Solution:**
We apply Gaussian elimination. We multiply the first row by $-2$ and 1 and add it to the second and third rows:
\[
\begin{pmatrix}
1 & -1 & 2 & 3 \\
2 & -2 & 5 & 1 \\
-1 & 1 & -5 & 12
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & -1 & 2 & 3 \\
0 & 0 & 1 & -5 \\
-3 & 15 & -3 & 15
\end{pmatrix}
\]

Now multiply the second row by 3 and add it to the third row:
\[
\begin{pmatrix}
1 & -1 & 2 & 3 \\
0 & 0 & 1 & -5 \\
0 & 0 & -3 & 15
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & -1 & 2 & 3 \\
0 & 0 & 1 & -5 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

(ii) Identify the basic variables and the free variables of the linear system of question.

**Solution:**
The pivots are in the first and third columns. So $x$ and $z$ are the basic variables and the remaining variables $y$ and $w$ are the free variables.
2. (20pts)
(i) Find the general solution to the linear equations in parametric form:

\[ \begin{align*}
   x - y + 2z + 3w &= 0 \\
   2x - 2y + 5z + w &= 0 \\
   -x + y - 5z + 12w &= 0 
\end{align*} \]

**Solution:** The coefficient matrix is the matrix \( A \) in question (1). We solve the equations from the echelon form using back substitution. By (1) (ii) \( y \) and \( w \) are the free variables.

\[ z - 5w = 0 \quad \text{so that} \quad z = 5w. \]

Therefore

\[ x - y + 2(5w) + 3w = 0 \quad \text{so that} \quad x = y - 13w. \]

The general form is

\[ (x, y, z, w) = (y - 13w, y, 5w, w) = y(1, 1, 0, 0) + w(-13, 0, 5, 1). \]

(ii) Check that \( (x, y, z, w) = (1, 1, -1, 1) \) is a solution to

\[ \begin{align*}
   x - y + 2z + 3w &= 1 \\
   2x - 2y + 5z + w &= -4 \\
   -x + y - 5z + 12w &= 17 
\end{align*} \]

**Solution:**

\[ \begin{align*}
   1 - 1 - 2 + 3 &= 1 \\
   2 - 2 - 5 + 1 &= -4 \\
   -1 + 1 + 5 + 12 &= 17. 
\end{align*} \]

(iii) Find the general solution to the linear equations, given in part (ii), in parametric form.

**Solution:**

The general solution is a sum of the particular solution \( (1, 1, -1, 1) \) and the general solution to the homogeneous:

\[ (x, y, z, w) = (1, 1, -1, 1) + y(1, 1, 0, 0) + w(-13, 0, 5, 1). \]
3. (10pts) (i) Show that the vectors \((1, 0, 0), (10, 2, 0)\) and \((-15, 3, 1)\) span \(\mathbb{R}^3\).

**Solution:**
Let \(A\) be the matrix whose columns are the vectors \((1, 0, 0), (10, 2, 0)\) and \((-15, 3, 1)\). We want to show that we can always solve the equation \(A\vec{x} = \vec{b}\). We apply elimination to
\[
A = \begin{pmatrix} 1 & 10 & -15 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 10 & -15 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1 \end{pmatrix}
\]
Every row contains a pivot. It follows that the augmented matrix contains no pivots in the last column and the equation \(A\vec{x} = \vec{b}\) is consistent. Therefore the vectors \((1, 0, 0), (10, 2, 0)\) and \((-15, 3, 1)\) span \(\mathbb{R}^3\).

4. (10pts) Find the values of \(h\) for which the vectors \((-1, 3, 2), (2, -6, -5)\) and \((1, h, 1)\) in \(\mathbb{R}^3\) are linearly dependent.

**Solution:** Let \(A\) be the matrix whose columns are the vectors \((-1, 3, 2), (2, -6, -5)\) and \((1, h, 1)\). We want to find those values of \(h\) such that \(A\vec{x} = \vec{0}\) has a non-trivial solution. We apply elimination to
\[
A = \begin{pmatrix} -1 & 2 & 1 \\ 3 & -6 & h \\ 2 & -5 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & -1 \\ 3 & -6 & h \\ 2 & -5 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & h + 3 \end{pmatrix}
\]
so that
\[
\rightarrow \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & h + 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & h + 3 \end{pmatrix}.
\]
There is a non-trivial solution if and only if there is a free variable. The only possibility is that the last variable is a free variable, that is, \(h + 3 = 0\). So the vectors \((-1, 3, 2), (2, -6, -5)\) and \((1, h, 1)\) in \(\mathbb{R}^3\) are linearly dependent if and only if \(h = -3\).
5. (15pts) Find the inverse of
\[
\begin{pmatrix}
1 & 2 & 3 \\
2 & 5 & 3 \\
1 & 0 & 8
\end{pmatrix}.
\]

Solution:
We apply Gauss-Jordan elimination. We first form the super augmented matrix
\[
\begin{pmatrix}
1 & 2 & 3 & 1 & 0 & 0 \\
2 & 5 & 3 & 0 & 1 & 0 \\
1 & 0 & 8 & 0 & 0 & 1
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 2 & 3 & 1 & 0 & 0 \\
0 & 1 & -3 & -2 & 1 & 0 \\
0 & -2 & 5 & -1 & 0 & 1
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 2 & 3 & 1 & 0 & 0 \\
0 & 1 & -3 & -2 & 1 & 0 \\
0 & 0 & -1 & -5 & -2 & 1
\end{pmatrix}
\]
Multiplying the last row by -1 completes the Gaussian elimination:
\[
\begin{pmatrix}
1 & 2 & 3 & 1 & 0 & 0 \\
0 & 1 & -3 & -2 & 1 & 0 \\
0 & 0 & 1 & 5 & -2 & -1
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 2 & 3 & 1 & 0 & 0 \\
0 & 1 & 0 & 13 & -5 & -3 \\
0 & 0 & 1 & 5 & -2 & -1
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 0 & 0 & -40 & 16 & 9 \\
0 & 1 & 0 & 13 & -5 & -3 \\
0 & 0 & 1 & 5 & -2 & -1
\end{pmatrix}
\]
The inverse is
\[
A^{-1} = \begin{pmatrix}
-40 & 16 & 9 \\
13 & -5 & -3 \\
5 & -2 & -1
\end{pmatrix}.
\]
6. (15pts) (i) Let \( f \) be the linear function
\[
 f : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \quad \text{given by} \quad (x, y, z) \mapsto (2x - y + 3z, x + 2y - z).
\]
Find a matrix \( A \) such that \( f(\vec{x}) = A\vec{x} \).

Solution: \( f(1, 0, 0) = (2, 1) \), \( f(0, 1, 0) = (-1, 2) \) and \( f(0, 0, 1) = (3, -1) \) and so
\[
 A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 2 & -1 \end{pmatrix}
\]

(ii) Let \( g \) be the linear function \( g : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) which represents rotation through \( \pi/2 \) about the origin. Find a matrix \( B \) such that \( g(\vec{x}) = B\vec{x} \).

Solution: \( g(1, 0) = (0, 1) \) and \( g(0, 1) = (-1, 0) \) and so
\[
 B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

(iii) Find a matrix \( C \) such that \( (g \circ f)(\vec{x}) = C\vec{x} \).

Solution: Composition of functions corresponds to matrix multiplication:
\[
 C = BA = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & 3 \\ 1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -2 & 1 \\ 2 & -1 & 3 \end{pmatrix}.
\]
7. (10pts) Let \( h \) be the linear function \( h: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) such that \( h(1, 1) = (5, 4) \) and \( h(1, 2) = (-1, 0) \). Find a matrix \( D \) such that \( h(\vec{x}) = D\vec{x} \).

Solution: \((0, 1) = (1, 2) - (1, 1)\). By linearity
\[
h(0, 1) = h(1, 2) - h(1, 1) = (-1, 0) - (5, 4) = (-6, -4).
\]

\((1, 0) = (1, 1) - (0, 1)\). By linearity
\[
h(1, 0) = h(1, 1) - h(0, 1) = (5, 4) - (-6, -4) = (11, 8).
\]

Therefore
\[
D = \begin{pmatrix} 11 & -6 \\ 8 & -4 \end{pmatrix}.
\]

We check:
\[
\begin{pmatrix} 11 & -6 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \text{ and } \begin{pmatrix} 11 & -6 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.
\]