SECOND MIDTERM
MATH 20F, UCSD, AUTUMN 14

You have 50 minutes. This test is closed book, closed notes, no calculators.

There are 6 problems, and the total number of points is 100. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:____________________________

Signature:__________________________

Student ID #:________________________

Dissertation instructor:________________

Dissertation Number+Time:______________

Problem	Points	Score
1	20	
2	15	
3	30	
4	15	
5	10	
6	10	
Total	100	

1. (20pts) Suppose that A and B are 3×3 matrices. If $\det A = 5$ and $\det B = -3$ what are
(i) $\det A^T$?

Solution: $\det A^T = \det A = 5$.

(ii) $\det A^{-1}$?

Solution: $\det A^{-1} = 1/\det A = 1/5$.

(iii) $\det AB$?

Solution: $\det AB = \det A \det B = -15$.

(iv) $\det 2A$?

Solution: $\det 2A = 2^3 \det A = 40$.

2. (15pts) Suppose that A is 9×16 matrix. Justify your answers to the following questions:
(i) What is the largest possible rank of A and the smallest possible dimension of the nullspace of A?

Solution: The maximum number of pivots is 9 and the rank is the number of pivots. So the rank is no larger than 9.
By rank-nullity the rank plus the nullity is 16. Therefore the smallest possible nullity is 7.

(ii) What is the largest possible rank of A^T and the smallest possible dimension of the nullspace of A^T?

Solution: The maximum number of pivots is 9 and the rank is the number of pivots. So the rank is no larger than 9.
By rank-nullity the rank plus the nullity is 9. Therefore the smallest possible nullity is 0.

(iii) Suppose that the set of $\vec{b} \in \mathbb{R}^9$ for which $A\vec{x} = \vec{b}$ has a solution is a subspace of dimension four. What is the dimension of the nullspace of A?

Solution: We are told the dimension of the column space is 4. Therefore the rank is 4. By rank-nullity the rank plus the nullity is 16. Therefore the nullity is 12.

3. (30pts) Consider the matrix

$$A = \begin{pmatrix} 4 & 4 & 0 & 4 \\ -3 & -6 & -3 & -9 \\ 0 & 2 & 2 & 4 \end{pmatrix}$$

(i) Find a basis for the nullspace of A.

Solution: We apply Gaussian elimination:

$$\to \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The elimination is complete.
There are pivots in the first and second rows and columns. x_1 and x_2 are basic variables, x_3 and x_4 are free variables.

$$x_2 + x_3 + 2x_4 = 0 \qquad \text{so that} \qquad x_2 = -x_3 - 2x_4.$$

Therefore

$$x_1 - x_3 - 2x_4 + x_4 = 0 \qquad \text{so that} \qquad x_1 = x_3 + x_4.$$

Thus

$$(x_1, x_2, x_3, x_4) = (x_3+x_4, -x_3-2x_4, x_3, x_4) = x_3(1, -1, 1, 0)+x_4(1, -2, 0, 1).$$

Therefore

$$\vec{n}_1 = (1, -1, 1, 0) \qquad \text{and} \qquad \vec{n}_2 = (1, -2, 0, 1)$$

is a basis for the nullspace.

(ii) Find a basis for the column space of A.

Solution: The pivots are in the first and second column. Therefore the first and second columns of A are a basis for the column space:

$$\vec{c}_1 = (4, -3, 0) \qquad \text{and} \qquad \vec{c}_2 = (4, -6, 2)$$

are a basis for the column space.

(iii) What is the rank of A?

Solution: 2.

(iv) Find a basis for the row space of A^T.

Solution: We apply Gaussian elimination to the transpose of A:

$$\begin{pmatrix} 4 & -3 & 0 \\ 4 & -6 & 2 \\ 0 & -3 & 2 \\ 4 & -9 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & -3/4 & 0 \\ 4 & -6 & 2 \\ 0 & -3 & 2 \\ 4 & -9 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & -3/4 & 0 \\ 0 & -3 & 2 \\ 0 & -3 & 2 \\ 0 & -6 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & -3/4 & 0 \\ 0 & 1 & -2/3 \\ 0 & -3 & 2 \\ 0 & -6 & 4 \end{pmatrix}$$

so that

$$\to \begin{pmatrix} 1 & -3/4 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The elimination is complete. The vectors

$$\vec{r}_1 = (1, -3/4, 0) \qquad \text{and} \qquad \vec{r}_2 = (0, 1, -2/3)$$

are a basis of the row space of A^T.

(v) Use the basis from part (iv) to find another basis for the column space of A.

Solution: The vectors

$$\vec{d}_1 = (1, -3/4, 0) \qquad \text{and} \qquad \vec{d}_2 = (0, 1, -2/3)$$

are a basis of the column space of A.

5. (15pts) Find the following determinant:

$$\begin{vmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & 4 & 4 \\ -1 & 0 & 1 & 0 \\ 2 & 0 & 4 & 1 \end{vmatrix}.$$

Solution:

$$\begin{aligned}
= -\begin{vmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 4 & 4 \\ 1 & 4 & 2 & 3 \\ 2 & 0 & 4 & 1 \end{vmatrix} &= \begin{vmatrix} 1 & 4 & 4 \\ 4 & 2 & 3 \\ 0 & 4 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 4 \\ 1 & 4 & 3 \\ 2 & 0 & 1 \end{vmatrix} \\
&= -\begin{vmatrix} 0 & 4 & 1 \\ 4 & 2 & 3 \\ 1 & 4 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} - 4\begin{vmatrix} 1 & 4 \\ 2 & 0 \end{vmatrix} \\
&= 4\begin{vmatrix} 4 & 3 \\ 1 & 4 \end{vmatrix} - \begin{vmatrix} 4 & 2 \\ 1 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} - 4\begin{vmatrix} 1 & 4 \\ 2 & 0 \end{vmatrix} \\
&= 4(4 \cdot 4 - 3) - (4 \cdot 4 - 2) - 5 - 4(-4 \cdot 2) \\
&= 52 - 14 - 5 + 32 \\
&= 65.
\end{aligned}$$

6. (10pts) For which scalars h are the vectors $\vec{v}_1 = (h, 1, 1)$, $\vec{v}_2 = (-2, -h, 3)$ and $\vec{v}_3 = (0, 1, -1)$ a basis of $\mathbb{R}^3$?

Solution: Let

$$A = \begin{pmatrix} h & -2 & 0 \\ 1 & -h & 1 \\ 1 & 3 & -1 \end{pmatrix}$$

be the matrix whose columns are the vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$.
Then $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are a basis if and only if A is invertible. A is invertible if and only if $\det A \neq 0$. We compute the determinant:

$$\begin{vmatrix} h & -2 & 0 \\ 1 & -h & 1 \\ 1 & 3 & -1 \end{vmatrix} = h \begin{vmatrix} -h & 1 \\ 3 & -1 \end{vmatrix} + 2 \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = h(h-3) + 2(-2) = h^2 - 3h - 4 = (h-4)(h+1).$$

The determinant is zero if and only if $h = -1$ or $h = 4$.
Thus $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are a basis if and only if h is equal to neither -1 nor 3.

7. (10pts) Are the following subsets of $\mathbb{R}^n$ linear subspaces? Explain your answer.
(i)

$$H = \{\, (a, b, c) \in \mathbb{R}^3 \,|\, a + b + c = -1 \,\}.$$

Solution: No. $\vec{0} \notin H$.

(ii)

$$H = \{\, (a, b, c, d) \in \mathbb{R}^3 \,|\, 3a + b = c \qquad a + b + 2c = 2d \,\}.$$

Solution: Yes. H is the set of solutions to the homogeneous linear equations:

$$\begin{aligned} 3a + b - c \qquad &= 0 \\ a + b + 2c - 2d &= 0. \end{aligned}$$

Thus H is the null space of the matrix

$$\begin{pmatrix} 3 & 1 & -1 & 0 \\ 1 & 1 & 2 & -2 \end{pmatrix}.$$