PRACTICE PROBLEMS FOR THE FIRST MIDTERM

Here are a slew of practice problems for the first midterm culled from old midterms:

1. Are the vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \qquad \vec{v}_3 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \qquad \text{and} \qquad \vec{v}_4 = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$$

linearly independent? If yes, explain why. If not find a non-trivial relation between these vectors.

2. Consider the matrix

$$A = \begin{pmatrix} 1 & -2 & 1 & 4 \\ 1 & -1 & 2 & 0 \\ -2 & 4 & 1 & 0 \end{pmatrix}.$$

(a) Find a matrix in echelon form which is row equivalent to A.
(b) Describe the solution set to the equation $A\vec{x} = \vec{0}$.
(c) Is there a vector $\vec{b} \in \mathbb{R}^3$ so that $A\vec{x} = \vec{b}$ has no solution? Justify your answer.

3. (a) Suppose a linear transformation $T\colon \mathbb{R}^5 \longrightarrow \mathbb{R}^6$ is one-to-one and expressed as $T(\vec{x}) = A\vec{x}$ for some matrix A. Determine the shape of A and the number of pivots.
(b) Suppose a linear transformation $T\colon \mathbb{R}^7 \longrightarrow \mathbb{R}^4$ is onto and expressed as $T(\vec{x}) = A\vec{x}$. Determine the shape of A and the number of pivots.
(c) Let

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 0 \\ 5 & 2 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 5 \\ -2 & 3 \end{pmatrix}$$

Which of the following products are defined? Calculate the ones that are defined.

$$AB, \qquad BA, \qquad A^2 \qquad \text{and} \qquad B^2.$$

4. Let $f(x_1, x_2, x_3) = (x_1 - 3x_2 + 2x_3, 2x_1 + 5x_2 - x_3)$. Find the matrix A associated to this function. Find a vector $\vec{x} = (x_1, x_2, x_3)$ such that $f(\vec{x}) = (1, 1)$.

5. Find the solution set S to the following equations:

$$\begin{aligned} x_1 + x_3 + x_4 &= 1 \\ 2x_2 + x_3 + x_4 &= 0 \\ x_1 + 2x_2 + x_3 &= 0. \end{aligned}$$

How many solutions are there?

6.

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

How many solutions does the homogeneous equation $A\vec{x} = \vec{0}$ have?

7. Do the vectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \qquad \text{and} \qquad \vec{v}_3 = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}$$

span $\mathbb{R}^3$?

8. Consider the matrix A and the vector $\vec{b}$ given by

$$\begin{pmatrix} 1 & -2 & 1 \\ 3 & 2 & 6 \\ 1 & 1 & -2 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}.$$

(a) Find the inverse of A. Justify your answer.

(b) Find the solution of the equation $A\vec{x} = \vec{b}$. Justify your answer.

9. Consider the matrices

$$A = \begin{pmatrix} 1 & 3 & -2 \\ 2 & 4 & -1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$$

Find a matrix C such that $BC = A$.

10. The echelon form of the matrix A is

$$\begin{pmatrix} 1 & * & * & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $*$'s denote arbitrary numbers.

(a) Does $A\vec{x} = \vec{0}$ have non-trivial solutions? You must give a reason to receive credit.

(b) Does $A\vec{x} = \vec{b}$ have at least one solution for every $\vec{b} \in \mathbb{R}^4$? You must give a reason to receive credit.

11. Write down the augmented matrix for the following linear equations and use it to find all solutions to the equations:

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 2 \\ 2x_1 + x_2 - 2x_3 &= 4 \\ x_1 - 4x_2 + 8x_3 &= 2. \end{aligned}$$

12. You need not give reasons in this problem
(a) For which values of p is it possible to find $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ so that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ span $\mathbb{R}^4$?
(b) For which values of p is it possible to find $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ so that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ are linearly independent?
13. Let A be the following matrix

$$\begin{pmatrix} 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \beta & 3 & 2 & 3 \\ 1 & 5 & 1 & 1 \end{pmatrix}.$$

(a) For which values of β is B invertible?
(b) Find all solutions to $A\vec{x} = \vec{0}$ in this case.
14. Describe the general solution to the system of equations

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 2 \\ -2x_1 - 2x_2 + 2x_3 &= 0 \\ 3x_1 + x_2 + x_2 &= 2 \end{aligned}$$

in parametric form.
15. Let C be the matrix

$$C = \begin{pmatrix} 1 & -1 & 3 \\ -2 & -2 & 2 \\ 3 & 1 & 1 \end{pmatrix}.$$

(a) Describe the general solution to the homogeneous equation $C\vec{x} = \vec{0}$.
(b) Does the system $C\vec{x} = \vec{b}$ have solutions for all possible choices of $\vec{b} \in \mathbb{R}^3$? Explain your answer.
16. Answer the following questions True or False. No explanations need be given.
(a) If A is any 5×7 matrix, then $A\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$.
(b) The columns of a 5×7 matrix always span $\mathbb{R}^5$.
(c) The columns of a 5×7 matrix are always linearly dependent.

17. Let A be the 4×3 matrix

$$A = \begin{pmatrix} 1 & -2 & k \\ 3 & h & 0 \\ 0 & 2 & 4 \\ 1 & 0 & 2 \end{pmatrix}.$$

Find all possible values of the parameters h and k for which the columns of A are linearly dependent.

18. Let $f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be a linear transformation such that

$$f(\vec{e}_1) = (3, -2) \qquad f(\vec{e}_2) = (3, 0) \qquad \text{and} \qquad f(\vec{e}_3) = (-3, 1).$$

Find the matrix associated to f.

19. Let

$$f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^4 \text{ given by } (x_1, x_2, x_3) \longrightarrow (-4x_2-4x_3, -2x_1+9x_2+5x_3, -x_1-2x_3, 3x_2+3x_3).$$

(a) Is f one-to-one? Why?

(b) Is f onto? Why?