PRACTICE PROBLEMS FOR THE SECOND MIDTERM

Here are a slew of practice problems for the second midterm culled from old midterms:

1. Let

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 4 & 5 \\ 2 & 8 & 11 \end{pmatrix}$$

Compute A^{-1}.

2. Let

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 1 & 3 & 5 \\ 0 & 2 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 4 & 0 & -1 \\ 0 & -3 & 4 \\ 0 & 4 & 3 \end{pmatrix}.$$

(a) Compute $\det A$.
(b) Compute $\det B$.
(c) Compute $\det AB$.
(d) Compute $\det A^T$.
(e) Which of A, B, AB and A^T are invertible?

3. Let

$$A = \begin{pmatrix} 2 & 4 & 6 \\ 2 & 5 & 8 \\ -2 & -3 & -4 \end{pmatrix}$$

(a) Find a basis for $\text{Col}(A)$.
(b) Find a basis for $\text{Row}(A)$.
(c) Find a basis for $\text{Nul}(A)$.

3. (a) If a 7×5 matrix A has rank 2 find:
(i) $\dim \text{Nul}(A)$.
(ii) $\text{rank}(A^T)$.
(b) If the null space of a 4×6 matrix A is 3-dimensional, what is $\dim \text{Col}(A)$?

4. The sets

$$\mathcal{B} = \{ (-1, 8), (1, -7) \} \qquad \text{and} \qquad \mathcal{C} = \{ (1, 2), (1, 1) \}$$

are both bases of $\mathbb{R}^2$.
Find the coordinates of the vectors in $\mathcal{B}$ in terms of the basis $\mathcal{C}$.

5. Let

$$A = \begin{pmatrix} 1 & -1 & 2 & 3 & 0 \\ 2 & -1 & 4 & 11 & 3 \\ -1 & 3 & -2 & 8 & 4 \\ 1 & 1 & 2 & 14 & 4 \end{pmatrix}.$$

(a) Find a basis for Col(A).
(b) Find a basis for Row(A).
(c) Find a basis for Nul(A).
6. Find the determinant of the matrix

$$\begin{pmatrix} 2 & 3 & -2 & 1 \\ 0 & 2 & 5 & 4 \\ 0 & -3 & 2 & -3 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

7. Let P_2 denote the space of polynomials of degree no greater than 2. Let

$$W = \{ p \in P_2 \,|\, p(-2) = 0 \}.$$

(a) Verify that H is a linear subspace of P_2.
(b) Give a careful definition of what is meant by a basis for a vector space.
(c) Find a basis for H. Justify your answer.
8. Let

$$A = \begin{pmatrix} 1 & 5 & -1 \\ 3 & 7 & -11 \\ -2 & -2 & 10 \end{pmatrix}.$$

Is the vector

$$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

in the column space of A? Justify your answer.
9. (a) If A is a 4×3 matrix, what is the largest dimension of the row space of A?
(b) If A is a 3×4 matrix, what is the largest dimension of the row space of A?
10. Let A be an $m \times n$ matrix. Show that the null space of A

$$\mathrm{Nul}\, A = \{ \vec{x} \in \mathbb{R}^n \,|\, A\vec{x} = \vec{0} \}$$

is closed under vector addition.
11. Suppose A is $n \times n$ and for some $\vec{b} \in \mathbb{R}^n$ the equation $A\vec{x} = \vec{b}$ has more than one solution. Can the columns of A span $\mathbb{R}^n$. Why or why not? Explain.
12. True or False:

(a) If A and B are two 3×3 matrices and

$$B = \begin{pmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{pmatrix}$$

then

$$AB = \left(A\vec{b}_1 + A\vec{b}_2 + A\vec{b}_3 \right).$$

(b) A plane in $\mathbb{R}^2$ is a two dimensional linear subspace of $\mathbb{R}^3$.
(c) If

$$\{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p \}$$

is a linearly independent set in a vector space V then $\dim V \geq p$.
(d) If $\vec{u}$ and $\vec{v}$ are two vectors in $\mathbb{R}^3$ then the rank of the matrix $\vec{u}\vec{v}^T$ is always 0 or 1.
(e) For a 3×3 matrix A, $\det(3A) = 3\det(A)$.
13. Let $f_0(t) = 1$, $f_1(t) = 1 + t$, $f_2(t) = 1 + t + t^2$, $f_3(t) = t^3$.
(a) Show that

$$\mathcal{B} = \{ f_0(t), f_1(t), f_2(t), f_3(t) \}$$

is a basis for the vector space P_3 of all polynomials of degree at most 3.
(b) Find the coordinates of the polynomial $f(t) = t^2 + t^3$ relative to $\mathcal{B}$.
14. Let

$$A = \begin{pmatrix} 2 & 1 & -1 & 0 & 3 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 & -1 \\ 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Compute $\det A$. Find a basis for the column space of A. What is the rank and the nullity of A?
15. If

$$f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^4$$

is a linear map and $\mathrm{Nul}(f) = \mathrm{Span}\{\vec{e}_1\}$, what is the dimension of the image of f?
16. Is

$$\mathcal{B} = \left\{ \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -6 \\ 11 \\ 10 \end{pmatrix} \right\}$$

a basis for $\mathbb{R}^3$?
17. Find b such that $(-1, b, 2, 3)$ is in the span of $(1, 2, 3, 4)$ and $(3, 4, 4, 5)$.
18. Can you give a simple reason why $\det(A) = 0$?

(a)

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -5 & 6 & 7 & 8 \\ -9 & 10 & 11 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(b)

$$A = \begin{pmatrix} 1 & 2 & 0 & 3 \\ -6 & 5 & 0 & 4 \\ -7 & 8 & 0 & 9 \\ 12 & 11 & 0 & 10 \end{pmatrix}.$$

(c)

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 10 & 20 & 30 & 40 \end{pmatrix}.$$

19.

$$A = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \beta & 3 & 2 & 3 \\ 1 & 5 & 1 & 1 \end{pmatrix}.$$

(a) For which values of β is A invertible?
(b) Asssuming A is singular then find the rank of A, the nullity of A and $\mathrm{Nul}(A)$.

20. Suppose that b_1, b_2, b_3 and b_4 are real numbers. Show that there is exactly one polynomial $p(t)$ in the vector space P_3 of polynomials of degree at most 3 such that:

$$p(1) = b_1, \qquad p'(0) = b_2, \qquad \int_{-1}^{1} p(t)\,\mathrm{d}t = b_3, \qquad \text{and} \qquad p(-1) = b_4.$$

21. Let

$$H = \mathrm{Span}\{\vec{u}, \vec{v}\} \qquad \text{and} \qquad K = \mathrm{Span}\{\vec{u}, \vec{v}, \vec{u} + \vec{v}\}.$$

Prove that $H = K$.

22. Suppose that $A^2 = 0$. Prove that A is not invertible.