13. Cauchys Integral Formula

Suppose that f is a holomorphic function, defined on a region U. Under what circumstances is it true that

$$\int f \,\mathrm{d}\gamma = 0,$$

for any closed path in U?

Note that this property will not hold in general. For example, take $f = 1/z$ and let U be the complement of the origin. Then if we integrate f, around a circle with centre 0, then the integral is equal to $2\pi i$ and certainly not zero:

Lemma 13.1. *Let R be a circle with centre a.*

$$\int_{\partial R} \frac{1}{z-a}\, dz = 2\pi i.$$

Proof. If we write $z = a + \rho e^{i\theta}$ then the integral becomes

$$\int_0^{2\pi} \frac{1}{\rho} e^{-i\theta}\, i\rho e^{i\theta}\, d\theta = \int_0^{2\pi} i\, d\theta = 2\pi i. \qquad \square$$

On the other hand, this property does hold on many regions. Note that if γ_1 and γ_2 are two paths with the same endpoints, then $\gamma = \gamma_1 - \gamma_2$ is a closed path. Thus the integral around γ is zero if and only if the integrals along γ_1 and γ_2 are the same. We start our investigation by considering this problem.

Suppose that we are given two functions p and q. Under what circumstances does the integral

$$\int_\gamma p\, dx + q\, dy$$

only depend on the endpoints?

Proposition 13.2. *Let p and q be two continuous functions, defined on a region U. The line integral*

$$\int_\gamma p\, \mathrm{d}x + q\, \mathrm{d}y$$

only depends on the endpoints of γ, for any path in U if and only if there is a function V such that

$$\frac{\partial V}{\partial x} = p \qquad \frac{\partial V}{\partial y} = q.$$

Proof. Suppose that there is such a function V. Then the line integral over γ may be computed as

$$\int_a^b \frac{\partial V}{\partial x} x'(t) + \frac{\partial V}{\partial y} y'(t)\, dt = \int_a^b \frac{dV}{dt}\,\mathrm{d}t = V(x(b), y(b)) - V(x(a), y(a)).$$

This expression clearly only depends on the endpoints of γ.

Now suppose that the line integral only depends on the endpoints. We may clearly assume that any two points of U are connected by a path. Pick a point of (x_0, y_0) of U. Define a function

$$V(x, y) = \int_\gamma p\, dx + q\, dy$$

where γ is any path that starts at (x_0, y_0) and ends at (x, y).

Now we can always arrange our path, so that the last segment is a horizontal straight line. The formula then becomes

$$V(x, y) = \int_\gamma p\, dx + \text{constant}$$

so that $\frac{\partial V}{\partial x} = p$. On the other hand arranging things so that the last segment is a vertical straight line, we get $\frac{\partial V}{\partial y} = q$. □

Now suppose that we look at

$$\int f(z)\,\mathrm{d}x + i f(z)\,\mathrm{d}y.$$

If the integral of this function only depends on the endpoints, then we have

$$\frac{\partial F}{\partial x} = f(z),$$

and

$$\frac{\partial F}{\partial y} = i f(z).$$

In this case F satisfies the Cauchy-Riemann equations

$$\frac{\partial F}{\partial x} = -i \frac{\partial F}{\partial y}.$$

Putting all this together, we get

Lemma 13.3. *Let $f(z)$ be a continuous function, defined on a region U.*

Then

$$\int f(z)\,\mathrm{d}\gamma$$

only depends on the endpoints of γ if and only if f is the derivative of a holomorphic function $F(z)$.

Theorem 13.4. *Let $f(z)$ be a function which is holomorphic on a circle.*

Then

$$\int f \, \mathrm{d}\gamma = 0$$

for any closed path in the circle.

Proof. Consider the integral

$$F(z) = \int f(z) \, \mathrm{d}\sigma,$$

where σ is the path from the centre to the point (x, y), which first varies x and then y. Then we have

$$\frac{\partial F}{\partial y} = i f(z).$$

By Cauchy's Theorem for a rectangle, we get exactly the same function, if we first vary y and then x, so that

$$\frac{\partial F}{\partial x} = f(z).$$

Now apply (13.2), to conclude that the integral around any path is zero. □

Lemma 13.5. *Let $f(z)$ be a function which is holomorphic outside finitely many points $a_1, a_2, \dots, a_k$ of a circle. In addition suppose that*

$$\lim_{z \to a_i} (z - a_i) f(z) = 0$$

for every i.

Then

$$\int f \, d\gamma = 0$$

for any closed path in the circle.

Proof. The proof is similar to the one given before. □

Theorem 13.6 (Cauchy's Integral Formula)**.** *Let γ be a circle with centre a and let $f(z)$ be a holomorphic function on the circle.*

Then

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - a} \, \mathrm{d}z$$

Proof. Consider the function

$$g(z) = \frac{f(z) - f(a)}{z - a}.$$

Then $g(z)$ satisfies the hypothesis of (13.5). Thus

$$0 = \int_\gamma g(z)\, dz = \int_\gamma \frac{f(z)}{z-a}\, dz - \int_\gamma \frac{f(a)}{z-a}\, dz.$$

But by (13.1) the latter integral is $2\pi i f(a)$. □