15. Consequences

Here are some consequences of Cauchy's integral formula:

Theorem 15.1 (Cauchy-Taylor)**.** *Let U be a region. Let a be any point of U.*

Then

$$
\begin{aligned}
f(z) &= f(a) + f'(a)(z-a) + f''(a)(z-a)^2/2 + \dots \\
&= f(a) + \frac{1}{2\pi i}\left(\int_\gamma \frac{f(w)}{(w-a)^2}\,\mathrm{d}w\right)(z-a) + \frac{1}{2\pi i}\left(\int_\gamma \frac{f(w)}{(w-a)^2}\,\mathrm{d}w\right)(z-a)^2 + \dots,
\end{aligned}
$$

where γ is any circle with centre a containing z and contained in U. Further the above power series is uniformly convergent in any disc contained in U.

In particular f is holomorphic on U if and only if it is analytic.

Proof. If f is analytic then we have already seen that it is holomorphic. Also we have already identified the derivative of an analytic function, so that we know the first displayed equality holds.

Thus it suffices to prove the given formula for a power series expression of $f(z)$. We start with Cauchy's Integral Formula,

$$f(z) = \frac{1}{2\pi i}\int_{\partial R} \frac{f(w)}{w-z}\,\mathrm{d}w.$$

Consider expanding

$$
\begin{aligned}
\frac{1}{w-z} &= \frac{1}{w-a+a-z} \\
&= \frac{1}{w-a}\frac{1}{1-\frac{z-a}{w-a}} \\
&= \frac{1}{w-a} + \frac{1}{(w-a)^2}(z-a) + \frac{1}{(w-a)^3}(z-a)^2 + \dots
\end{aligned}
$$

As this is a geometric series, this is uniformly convergent, for

$$|z-a| < (1-\epsilon)|w-a|,$$

and for any $\epsilon > 0$. But then

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{w-z}\,\mathrm{d}w \\
&= \frac{1}{2\pi i}\int_\gamma \left(\frac{f(w)}{w-a} + \frac{f(w)}{(w-a)^2}(z-a) + \frac{f(w)}{(w-a)^3}(z-a)^2 + \dots\right)\mathrm{d}w \\
&= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-a}\,\mathrm{d}w + \left(\frac{1}{2\pi i}\int \frac{f(w)}{(w-a)^2}\,\mathrm{d}w\right)(z-a) + \dots,
\end{aligned}
$$

where we are allowed to switch the order of summation and integration by uniform convergence. □

Corollary 15.2 (Cauchy's derivative formula)**.** *Let U be a region and let γ be a circle with centre a, completely contained in U. Let $f(z)$ be a holomorphic function on U and let z be a point inside the circle.*

Then

$$f^n(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^{n+1}}\,\mathrm{d}w.$$

Proof. If $z = a$ is the centre of the circle this is immediate from (15.1). The general result follows as in the proof of (15.1), replacing a circle centred at a with the circle γ. □

Theorem 15.3 (Morera's Theorem)**.** *Let $f(z)$ be a continuous function, defined on a region U.*

If

$$\int_{\partial R} f(z)\,\mathrm{d}z = 0,$$

for all rectangles R contained in the region U then $f(z)$ is holomorphic.

Proof. (13.3) implies that there is a holomorphic function $F(z)$ such that $F'(z) = f(z)$. By (15.1) $F(z)$ is analytic. But then $f(z)$ is analytic and so $f(z)$ is holomorphic. □

Proposition 15.4 (Cauchy's Inequality)**.** *Let $f(z) = \sum_n a_n(z-a)^n$ be the power series expansion of a holomorphic function in a disc Δ with centre a.*

Then

$$|a_n| \leq \frac{M(r)}{r^n},$$

where $M(r)$ is the maximum value of $|f(z)|$ over a circle with centre a and radius r.

Proof. By (15.2),

$$\begin{aligned} |a_n| &= \left|\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^{n+1}}\,\mathrm{d}w\right| \\ &\leq \frac{1}{2\pi}\frac{M(r)}{r^{n+1}}2\pi r \\ &= \frac{M(r)}{r^n}. \end{aligned}$$

□

Theorem 15.5 (Liouville's Theorem)**.** *Every bounded entire holomorphic function $f(z)$ is constant.*

Proof. We give two proofs. Let M be any real number such that $|f(z)| \leq M$.

As $f(z)$ is entire, by Cauchy-Taylor, we have a power series expansion for $f(z)$ with infinite radius of convergence,

$$f(z) = \sum_n a_n z^n.$$

By Cauchy's estimate

$$|a_n| \leq \frac{M}{r^n}.$$

As this is independent of r, $a_n = 0$ if $n > 0$ and $f(z)$ is constant.

Aliter: Let z_1 and z_2 be two points in the plane. Then

$$\begin{aligned} f(z_1) - f(z_2) &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w - z_1}\,\mathrm{d}w - \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w - z_2}\,\mathrm{d}w \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)(z_1 - z_2)}{(w - z_1)(w - z_2)}\,\mathrm{d}w, \end{aligned}$$

where γ is any circle big enough to contain both z_1 and z_2.

If the radius of the circle is R then

$$|f(z_1) - f(z_2)| \leq \frac{MR|z_1 - z_2|}{(R - |z_1|)(R - |z_2|)}.$$

As R gets larger, the RHS tends to zero. It follows then that $f(z_1) = f(z_2)$ and so f is constant. □

Corollary 15.6 (Fundamental Theorem of Algebra)**.** *Let $P(z)$ be a polynomial of degree n with complex coefficients.*

Then $P(z)$ has n roots.

Proof. We may assume that $n > 0$ and it suffices to prove that $P(z)$ has at least one root. We may assume that $P(z)$ is monic.

Suppose not. Let $f(z) = \frac{1}{P(z)}$. Then f is an entire function.

On the other hand, let M be the maximum of the coefficients of $P(z)$, other than the leading coefficient. Then

$$|P(z) - z^n| \leq nM|z|^{n-1},$$

so that if $|z| > 2nM$ then

$$|P(z)| > |z|.$$

Thus

$$|f(z)| = \frac{1}{|P(z)|},$$

tends to zero as $|z|$ tends to infinity. Thus f is a bounded entire holomorphic function and so f is constant, a contradiction. □