16. Zeroes and Poles Revisited

Definition-Lemma 16.1. *Let f be a holomorphic function, defined on a region U.*

*The **order of vanishing** of f at $a \in U$ is the largest n such that*

$$f^i(a) = 0 \qquad \text{for all } i < n.$$

If f has order of vanishing n then we may write

$$f(z) = (z-a)^n g(z),$$

where $g(z)$ is holomorphic in a neighbourhood of a, and $g(a) \neq 0$.

Proof. As f is holomorphic it is analytic and so we may write

$$f(z) = \sum_{m \in \mathbb{N}} a_m (z-a)^m$$

as a power series, valid in some neighbourhood of a. $f^i(a) = 0$ implies that $a_i = 0$ so that

$$f(z) = \sum_{m \geq n} a_m (z-a)^m = (z-a)^n \sum_{m \in \mathbb{N}} a_{m+n} (z-a)^m = (z-a)^n g(z),$$

where

$$g(z) = \sum_{m \in \mathbb{N}} a_{m+n} (z-a)^m$$

is a power series. It is easy to see that g has the same radius of convergence as f so that g is analytic. But then g is holomorphic. □

Lemma 16.2. *Let $f(z)$ and $g(z)$ be two holomorphic functions on U. Suppose that there is a sequence of points $a_1, a_2, \dots$, with a limit point a in U, such that $f(a_i) = g(a_i)$.*

Then $f = g$ on U.

Proof. Looking at the difference of f and g, it suffices to prove that any function f, which is zero on a set of points $a_1, a_2, \dots$ in U, with an accumulation point a in U, must be identically zero.

Suppose not. By (16.1) we may write $f(z) = (z-a)^n g(z)$, where $g(a) \neq 0$. By continuity, there is an open neighbourhood V of a such that $g(z) \neq 0$ on V. But then $f(z) \neq 0$, in $V - \{a\}$, a contradiction. □

Definition 16.3. *Let U be a region, a a point of U and f a function holomorphic on $U - \{a\}$.*

*We say that $f(z)$ has a **removable singularity** at a if there is a holomorphic function $g(z)$ on U, such that $f = g$ on $U - \{a\}$.*

Lemma 16.4. *Let U be a region and let f be a holomorphic function on $U - \{a\}$, where $a \in U$.*

a is a removable singularity of $f(z)$ if and only if

$$\lim_{z \to a} (z-a) f(z) = 0.$$

Proof. One direction is clear. If $f(z)$ has a removable singularity at a then

$$\lim_{z \to a} (z-a) f(z) = \lim_{z \to a} (z-a) g(z) = 0.$$

For the other direction, pick a disc contained in U centred at a. Let γ be the boundary of the disc. Define a function $g(z)$ on the disc by the formula

$$g(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} \, \mathrm{d}w.$$

By Cauchy's Integral Formula, $f = g$ outside a. Consider integrating g around a small rectangle R in the disc.

$$\begin{aligned} \int_{\partial R} g(z) \, \mathrm{d}z &= \int_{\partial R} \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} \, \mathrm{d}w \, \mathrm{d}z \\ &= \int_{\gamma} f(w) \frac{1}{2\pi i} \int_{\partial R} \frac{1}{w - z} \, \mathrm{d}z \, \mathrm{d}w \\ &= -n(\partial R; w) \int_{\gamma} f(w) \, \mathrm{d}w \\ &= 0, \end{aligned}$$

and so g is holomorphic by Morera's Theorem. □

Theorem 16.5. *Let f be a holomorphic function on a region U. Let γ be a closed path in U. Suppose that $n(\gamma; w) = 0$ for all $w \in \mathbb{C} - U$.*

Then for every $a \in U - \gamma$,

$$n(\gamma; a) f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - a} \, \mathrm{d}z.$$

Proof. Define a function $\phi \colon U \times U \longrightarrow \mathbb{C}$ as follows;

$$\phi(z, w) = \begin{cases} \frac{f(z) - f(w)}{z - w} & z \neq w \\ f'(z) & z = w. \end{cases}$$

Clearly ϕ is continuous and the function $z \longrightarrow \phi(z, w)$ is holomorphic, for any fixed $w \in U$, since $z = w$ is a removable singularity.

Let

$$V = \{ \, w \in \mathbb{C} - \gamma \mid n(\gamma; w) = 0 \, \}.$$

Then V is open and $U \cup V = \mathbb{C}$, by hypothesis. Look at the function

$$g(z) = \begin{cases} \int_\gamma \phi(z,w)\,\mathrm{d}w & z \in U \\ \int_\gamma \frac{f(w)}{w-z}\,\mathrm{d}w & z \in V. \end{cases}$$

Observe that

$$\int_\gamma \frac{f(w)}{w-z}\,\mathrm{d}w$$

is a holomorphic function. For example, one can expand

$$\frac{1}{w-z}$$

as a power series in z and integrate term by term. Or one argue as in the proof of (16.4).

Note also that

$$\begin{aligned} \int_\gamma \phi(z,w)\,\mathrm{d}w &= \int_\gamma \frac{f(z)-f(w)}{z-w}\,\mathrm{d}w \\ &= \int_\gamma \frac{f(w)}{w-z}\,\mathrm{d}w - f(z)\int_\gamma \frac{1}{w-z}\,\mathrm{d}w \\ &= \int_\gamma \frac{f(w)}{w-z}\,\mathrm{d}w - n(\gamma;z)f(z). \end{aligned}$$

Thus

$$\int_\gamma \phi(z,w)\,\mathrm{d}w$$

is holomorphic. If $z \in U \cap V$ then

$$\int_\gamma \phi(z,w)\,\mathrm{d}w = \int_\gamma \frac{f(w)}{w-z}\,\mathrm{d}w$$

since $n(\gamma;w) = 0$ by definition of V. Thus $g(z)$ is a well-defined entire function.

It suffices to prove that $g(z) = 0$. Consider what happens as z tends to infinity. Then $\frac{1}{|z-w|}$ tends uniformly to zero as z tends to infinity, where w is on γ. Thus $g(z)$ tends to zero. As g is entire and bounded, by Liouville it is constant and as it tends to zero, in fact it is zero. □

Definition 16.6. *Let U be a region and let f be a function holomorphic on $U - \{a\}$, where $a \in U$. Then a is called an* ***isolated singularity*** *of f.*

We say that f has a ***pole, of order*** n***,*** *at a, if*

$$(z-a)^n f(z)$$

has a removable singularity at a, but $(z-a)^m f(z)$ does not, for any $m < n$. If there is no such n then we say that f has an ***essential singularity*** *at a.*

There are other ways to define a pole and to define the order:

Lemma 16.7. *Let U be a region and let f be a function holomorphic on $U - \{a\}$, where $a \in U$.*

f has a pole at a if and only if $\lim_{z \to a} f(z) = \infty$. In this case the order of the pole of $f(z)$ at $z = a$ is equal to the order of the zero of $g(z)$, where $g(z) = \frac{1}{f(z)}$.

Proof. If f has a pole of order n at a then we can find h holomorphic such that

$$f(z) = \frac{h(z)}{(z-a)^n},$$

where $h(z) \neq 0$ and so

$$\lim_{z \to a} f(z) = \lim_{z \to a} \frac{h(z)}{(z-a)^n} = \infty.$$

In this case

$$g(z) = \frac{(z-a)^n}{h(z)}$$

has a zero of order n.

Now suppose that

$$\lim_{z \to a} f(z) = \infty.$$

Then there is a neighbourhood of a where f is not zero. Thus $g(z)$ is holomorphic in a punctured neighbourhood of a. By assumption

$$\lim_{z \to a} g(z) = 0,$$

so that g has a removable singularity. Thus $g(z) = (z-a)^n h(z)$ for some holomorphic function h and so f has a pole at a. □

As usual we can extend the definition of zeroes and poles to the extended complex plane. A zero of $f(z)$ at $z = \infty$ is zero of $g(z) = f(1/z)$. A pole of $f(z)$ at ∞ is a pole of $g(z) = f(1/z)$, that is, a zero of $1/g(z) = 1/(f(1/z))$.

Definition 16.8. *Let f be a holomorphic function, with isolated singularities on a region U.*

We say that f is ***meromorphic*** *on U if the singularities of f are all poles.*

In a sense to be made precise later, a meromorphic function is a holomorphic function $U \longrightarrow \mathbb{P}^1$.

More interestingly we can define the order of a function $f(z)$ as a real number α.

Lemma 16.9. *Let $f(z)$ be a meromorphic function with a pole at a.*

Then the order of f is exactly equal to the largest real number α such that the limit

$$\lim_{z \to a} |f(z)||z-a|^{\alpha}$$

exists.

Proof. Suppose that the order of f at a is n. By definition of α then

$$\alpha \leq n.$$

Suppose that $\alpha < n$. Then

$$\lim_{z \to a} |f(z)||z-a|^{n} = \lim_{z \to a} (|f(z)||z-a|^{\alpha}) \lim_{z \to a} |z-a|^{n-\alpha} = 0,$$

which contradicts the definition of the order. □

Lemma 16.10. *Let $f(z)$ be a meromorphic function on U, with a pole of order n at a.*

Then there is a series expansion for $f(z)$, valid in a punctured neighbourhood of $z = a$,

$$f(z) = \sum_{k=-n} b_k (z-a)^k,$$

where $b_{-n} \neq 0$.

Proof. Clear, applying Cauchy-Taylor to $g(z) = (z-a)^n f(z)$. □

Theorem 16.11 (Casorati-Weierstrass)**.** *Suppose that f is a holomorphic function on $U - \{a\}$, with an essential singularity at the point a of the region U.*

Then, for every w, $\delta > 0$ and $\epsilon > 0$, there are infinitely many points z, such that $|z-a| < \delta$ and $|f(z) - w| < \epsilon$.

Proof. Given w, δ and ϵ, it suffices to prove that there is one such point z.

Suppose not. Consider

$$g(z) = \frac{f(z) - w}{z - a}.$$

Then $\lim_{z \to a} g(z) = \infty$. It follows, by (16.7), that $g(z)$ has a pole at $z = a$, say of order n. Then

$$g(z) = \frac{h(z)}{(z-a)^n},$$

where $h(z)$ is holomorphic in a neighbourhood of a. But then

$$f(z) = \frac{h(z)}{(z-a)^{n-1}} + w,$$

visibly meromorphic in a neighbourhood of a, a contradiction. □

In other words, in any neighbourhood of an essentially singularity, a function gets arbitrarily close to any complex number.