17. Counting Zeroes and the Open Mapping Theorem

Let \( f \) be a holomorphic function such that \( f(a) = b \). We say that the order of \( f \) at \( a \) is is the order of \( f - b \), that is, the largest \( n \) such that

\[
f^i(a) = 0 \quad \text{for all} \quad 0 < i < n.
\]

**Theorem 17.1.** Let \( f(z) \) be a non-constant holomorphic function on a disc \( \Delta \). Suppose that \( a \in \mathbb{C} \) and let \( z_1, z_2, \ldots \) be the complex numbers, with repetition according to the order, such that \( f(z) = a \). Then

\[
\sum_j n(\gamma, z_j) = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z) - a} \, dz,
\]

where \( \gamma \) is any closed curve in \( \Delta \), not containing any of the points \( z_1, z_2, \ldots \) and where the sum has only finitely many terms.

**Proof.** Replacing \( f \) by \( f - a \) we may assume that \( a = 0 \), so that \( z_1, z_2, \ldots \) are the zeroes of \( f \).

Since the only accumulation points of \( z_1, z_2, \ldots \) are on the boundary of \( \Delta \), possibly replacing \( \Delta \) by a smaller disc we may assume that \( f \) has only finitely many zeroes in \( \Delta \). By definition of the order of a zero and an obvious induction, we may write

\[f(z) = (z - z_1)(z - z_2)(z - z_3) \cdots (z - z_k)g(z),\]

where, as pointed out above, we allow repetition. By assumption \( g(z) \) is a holomorphic function on \( \Delta \) with no zeroes \( z \) in \( \Delta \). We have

\[
\frac{f'(z)}{f(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \cdots + \frac{1}{z - z_k} + \frac{g'(z)}{g(z)}.
\]

On the other hand

\[
\int_{\gamma} \frac{g'(z)}{g(z)} \, dz = 0,
\]

by Cauchy’s Theorem, since . The result follows by integrating both sides around \( \gamma \) and applying the definition of the winding number. \( \Box \)

The formula above reads more concisely as

**Corollary 17.2.**

\[n(\Gamma, a) = \sum n(\gamma, z_j).\]

**Proof.** Pick a parametrisation of

\[
\gamma: [c, d] \rightarrow \mathbb{C}.
\]

Then \( \Gamma \) has the parametrisation

\[
\Gamma: [c, d] \rightarrow \mathbb{C} \quad \text{given by} \quad \Gamma(t) = f(\gamma(t)).
\]
By definition of the path integral:
\[
\int_{\Gamma} \frac{1}{w-a} \, dw = \int_{c}^{d} \frac{\Gamma'(t)}{\Gamma(t)-a} \, dt \\
= \int_{c}^{d} \frac{f'(\gamma(t))\gamma'(t)}{f(\gamma(t))-a} \, dt \\
= \int_{\gamma} \frac{f'(z)}{f(z)-a} \, dz. 
\]

**Theorem 17.3.** Let \( f \) be a function, holomorphic in a neighbourhood of \( z_0 \) and set \( w_0 = f(z_0) \).

If \( f(z) - w_0 \) has a zero of order \( n \) at \( z_0 \) then there is a \( \delta > 0 \) and an \( \epsilon > 0 \), such that for all \( 0 < |a - w_0| < \epsilon \) the equation
\[
f(z) = a \quad \text{subject to} \quad |z - z_0| < \delta
\]
has \( n \) roots.

**Proof.** Note that \( f' \) is a holomorphic function, so that its zeroes are isolated. Pick \( \delta > 0 \) so that \( f(z) \) is holomorphic, \( f'(z) \) is only zero if \( z = z_0 \) and the equation
\[
f(z) = w_0,
\]
has only the root \( z = z_0 \), for \( |z - z_0| < \delta \). Let \( \gamma \) be the circle \( |z - z_0| = \delta \) and let \( \Gamma \) be the image of \( \gamma \). As \( w_0 \notin \Gamma \) and \( \Gamma \) is closed, we may find \( \epsilon > 0 \) so that \( |w - w_0| < \epsilon \) does not intersect \( \Gamma \).

Suppose that \( 0 < |a - w_0| < \epsilon \). Then \( a \) and \( w_0 \) belong to the same connected component of \( \mathbb{C} - \Gamma \). Therefore
\[
n(\Gamma; a) = n(\Gamma; w_0). 
\]

As \( f(z) - w_0 \) has a zero of order \( n \), (17.2) implies that the RHS is \( n \). As \( f'(z) \neq 0 \) for \( z \neq z_0 \), then the order of zero of any solution to \( f(z) = a \) is one. The result then follows by (17.2). \( \square \)

**Definition 17.4.** Let \( f : X \longrightarrow Y \) be a map of topological spaces. \( f \) is called **open** if the image of every open set is open.

**Theorem 17.5.** (Open Mapping Theorem) Every holomorphic map is open.

**Proof.** As every open subset is a union of open balls, it suffices to prove that the image of a sufficiently small open ball is a union of open balls. This follows from (17.3). \( \square \)

**Corollary 17.6.** Let \( f(z) \) be holomorphic at \( z = z_0 \) and suppose that \( f'(z_0) \neq 0 \).

Then \( f \) is a local homeomorphism and locally conformal.
Proof. By (17.3) \( f \) is locally a bijection. Since it is also open and continuous, it is automatically a homeomorphism.