17. Counting Zeroes and the Open Mapping Theorem

Let f be a holomorphic function such that $f(a) = b$. We say that the order of f at a is is the order of $f - b$, that is, the largest n such that

$$f^i(a) = 0 \qquad \text{for all} \qquad 0 < i < n.$$

Theorem 17.1. *Let $f(z)$ be a non-constant holomorphic function on a disc Δ. Suppose that $a \in \mathbb{C}$ and let $z_1, z_2, \dots$ be the complex numbers, with repetition according to the order, such that $f(z) = a$. Then*

$$\sum_j n(\gamma, z_j) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z) - a}\,\mathrm{d}z,$$

where γ is any closed curve in Δ, not containing any of the points $z_1, z_2, \dots$ and where the sum has only finitely many terms.

Proof. Replacing f by $f - a$ we may assume that $a = 0$, so that $z_1, z_2, \dots$ are the zeroes of f.

Since the only accumulation points of $z_1, z_2, \dots$ are on the boundary of Δ, possibly replacing Δ by a smaller disc we may assume that f has only finitely many zeroes in Δ. By definition of the order of a zero and an obvious induction, we may write

$$f(z) = (z - z_1)(z - z_2)(z - z_3)\dots(z - z_k)g(z),$$

where, as pointed out above, we allow repetition. By assumption $g(z)$ is a holomorphic function on Δ with no zeroes z in Δ. We have

$$\frac{f'(z)}{f(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \dots + \frac{1}{z - z_k} + \frac{g'(z)}{g(z)}.$$

On the other hand

$$\int_\gamma \frac{g'(z)}{g(z)}\,\mathrm{d}z = 0,$$

by Cauchy's Theorem, since . The result follows by integrating both sides around γ and applying the definition of the winding number. □

The formula above reads more concisely as

Corollary 17.2.

$$n(\Gamma, a) = \sum n(\gamma, z_j).$$

Proof. Pick a parametrisation of

$$\gamma \colon [c, d] \longrightarrow \mathbb{C}.$$

Then Γ has the parametrisation

$$\Gamma \colon [c, d] \longrightarrow \mathbb{C} \qquad \text{given by} \qquad \Gamma(t) = f(\gamma(t)).$$

By definition of the path integral:

$$\begin{aligned}\int_\Gamma \frac{1}{w-a}\,\mathrm{d}w &= \int_c^d \frac{\Gamma'(t)}{\Gamma(t)-a}\,\mathrm{d}t \\ &= \int_c^d \frac{f'(\gamma(t))\gamma'(t)}{f(\gamma(t))-a}\,\mathrm{d}t \\ &= \int_\gamma \frac{f'(z)}{f(z)-a}\,\mathrm{d}z. \qquad \square\end{aligned}$$

Theorem 17.3. *Let f be a function, holomorphic in a neighbourhood of z_0 and set $w_0 = f(z_0)$.*

If $f(z) - w_0$ has a zero of order n at z_0 then there is a $\delta > 0$ and an $\epsilon > 0$, such that for all $0 < |a - w_0| < \epsilon$ the equation

$$f(z) = a \qquad \textit{subject to} \qquad |z - z_0| < \delta$$

has n roots.

Proof. Note that f' is a holomorphic function, so that its zeroes are isolated. Pick $\delta > 0$ so that $f(z)$ is holomorphic, $f'(z)$ is only zero if $z = z_0$ and the equation

$$f(z) = w_0,$$

has only the root $z = z_0$, for $|z - z_0| < \delta$. Let γ be the circle $|z - z_0| = \delta$ and let Γ be the image of γ. As $w_0 \notin \Gamma$ and Γ is closed, we may find $\epsilon > 0$ so that $|w - w_0| < \epsilon$ does not intersect Γ.

Suppose that $0 < |a - w_0| < \epsilon$. Then a and w_0 belong to the same connected component of $\mathbb{C} - \Gamma$. Therefore

$$n(\Gamma; a) = n(\Gamma; w_0).$$

As $f(z) - w_0$ has a zero of order n, (17.2) implies that the RHS is n. As $f'(z) \neq 0$ for $z \neq z_0$, then the order of zero of any solution to $f(z) = a$ is one. The result then follows by (17.2). $\square$

Definition 17.4. *Let $f \colon X \longrightarrow Y$ be a map of topological spaces. f is called* **open** *if the image of every open set is open.*

Theorem 17.5. *(Open Mapping Theorem) Every holomorphic map is open.*

Proof. As every open subset is a union of open balls, it suffices to prove that the image of a sufficiently small open ball is a union of open balls. This follows from (17.3). $\square$

Corollary 17.6. *Let $f(z)$ be holomorphic at $z = z_0$ and suppose that $f'(z_0) \neq 0$.*

Then f is a local homeomorphism and locally conformal.

Proof. By (17.3) f is locally a bijection. Since it is also open and continuous, it is automatically a homeomorphism. □