18. The maximum principle

Theorem 18.1 (Maximum principle)**.** *If $f(z)$ is holomorphic and non-constant on a region U then $|f(z)|$ does not achieve its maximum on U.*

Proof. Suppose that $f(z_0) = w_0$. Then there exists $\delta > 0$ and $\epsilon > 0$ such that for all $|z - z_0| < \delta$ and $|w - w_0| < \epsilon$, the equation

$$f(z) = w,$$

has at least one solution. Pick w such that $|w - w_0| < \epsilon$ and yet $|w| > |w_0|$. Pick z such that $f(z) = w$. Then $|f(z)| > |f(z_0)|$. Thus z_0 is not a maximum of $|f|$. As z_0 is arbitrary we are done. □

There is another way to state this result.

Theorem 18.2 (Maximum principle: bis)**.** *If $f(z)$ is defined on a closed and bounded set E and f is holomorphic on the interior, then $|f|$ achieves its maximum on the boundary of E.*

Proof. Since E is closed and bounded, it is compact. As f is continuous on E, it follows that $|f|$ achieves it maximum somewhere on E. If f is constant there is nothing to prove. Otherwise by (18.1) this point is not a point of the interior of E. □

Even though the maximum principle is an easy consequence of the open mapping Theorem, it is interesting to give a direct proof of this result.

Proof of (18.1). We give two proofs. Suppose that $z_0 \in U$ is a point where $|f|$ achieves its maximum.

First proof: Pick a small circle γ about z_0 and apply Cauchy's Integral Formula

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} \, \mathrm{d}z \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z) \, \mathrm{d}\theta. \end{aligned}$$

Taking absolute values, and noting that $|f(z)| \leq |f(z_0)|$, by (18.3) we see that $|f(z)| = |f(z_0)|$. It follows that $|f(z)|$ is constant on the whole of U, so that $f(z)$ is constant.

Second proof: As f is holomorphic, we may find a power series representation of f about z_0,

$$f(z) = \sum_n a_n (z - z_0)^n.$$

Let $z = z_0 + re^{i\theta}$. Then

$$f(z) = \sum_n a_n r^n e^{in\theta}.$$

$$|f(z)|^2 = f(z)\bar{f}(z) = \sum_{m,n\in\mathbb{N}} \bar{a}_m a_n r^{m+n} e^{i(n-m)\theta}.$$

Hence

$$\begin{aligned}\frac{1}{2\pi}\int_\gamma |f(z)|^2\,\mathrm{d}z &= \frac{1}{2\pi}\sum_{m,n\in\mathbb{N}}\int_0^{2\pi} \bar{a}_m a_n r^{n+m} e^{i(n-m)\theta}\,\mathrm{d}\theta \\ &= \sum_{n,m\in\mathbb{N}} |a_n|^2 r^{2n}.\end{aligned}$$

But the integral on the LHS is no more than $|a_0|^2$, by (18.3). It follows that $a_n = 0$, for all $n \geq 1$. Thus f is constant. □

Lemma 18.3. *Let $\phi(x)$ be a continuous function such that $\phi(x) \leq k$.*

Then

$$\frac{1}{b-a}\int_a^b \phi(x)\,\mathrm{d}x \leq k,$$

with equality if and only if $\phi(x) = k$.

Proof. Easy. □

Lemma 18.4. *Let u be a non-constant harmonic function defined on a region U.*

Then u does not achieve its maximum on U.

Proof. Note that if u is not constant then it is not locally constant. So it is enough to prove this locally. Thus we may assume that u has a harmonic conjugate v, so that u is the real part of a holomorphic function f. Then e^f is holomorphic and $|e^f| = |e^u|$. □

Lemma 18.5 (Schwarz's Lemma)**.** *Let f be a holomorphic function from the interior of the unit disc to the unit disc.*

If $f(0) = 0$, then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. Furthermore if either $|f'(0)| = 1$ of $|f(z)| = |z|$ for some $z \neq 0$ then $f(z) = cz$, for some constant c, of absolute value one.

Proof. Note that

$$g(z) = \begin{cases} \frac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0. \end{cases}$$

has a removable singularity at 0. Thus we may apply the maximum principle to $g(z)$.

On a circle of radius r, $|g(z)| \leq 1/r$. Thus $|g(z)| \leq 1/r$ on the circle $|z| \leq r$ by the maximum principle. Letting r tend to one, we see that $|g(z)| \leq 1$. By the maximum principle $|g(z)| < 1$ for all z unless it is constant and this gives the result. $\square$