18. The maximum principle

**Theorem 18.1** (Maximum principle). If \( f(z) \) is holomorphic and non-constant on a region \( U \) then \( |f(z)| \) does not achieve its maximum on \( U \).

*Proof.* Suppose that \( f(z_0) = w_0 \). Then there exists \( \delta > 0 \) and \( \epsilon > 0 \) such that for all \( |z - z_0| < \delta \) and \( |w - w_0| < \epsilon \), the equation

\[
f(z) = w,
\]

has at least one solution. Pick \( w \) such that \( |w - w_0| < \epsilon \) and yet \( |w| > |w_0| \). Pick \( z \) such that \( f(z) = w \). Then \( |f(z)| > |f(z_0)| \). Thus \( z_0 \) is not a maximum of \( |f| \). As \( z_0 \) is arbitrary we are done. \( \square \)

There is another way to state this result.

**Theorem 18.2** (Maximum principle: bis). If \( f(z) \) is defined on a closed and bounded set \( E \) and \( f \) is holomorphic on the interior, then \( |f| \) achieves its maximum on the boundary of \( E \).

*Proof.* Since \( E \) is closed and bounded, it is compact. As \( f \) is continuous on \( E \), it follows that \( |f| \) achieves its maximum somewhere on \( E \). If \( f \) is constant there is nothing to prove. Otherwise by (18.1) this point is not a maximum of \( |f| \). As \( z_0 \) is arbitrary we are done. \( \square \)

Even though the maximum principle is an easy consequence of the open mapping Theorem, it is interesting to give a direct proof of this result.

*Proof of* (18.1). We give two proofs. Suppose that \( z_0 \in U \) is a point where \( |f| \) achieves its maximum.

*First proof:* Pick a small circle \( \gamma \) about \( z_0 \) and apply Cauchy’s Integral Formula

\[
f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0} \, dz
= \frac{1}{2\pi} \int_0^{2\pi} f(z) \, d\theta.
\]

Taking absolute values, and noting that \( |f(z)| \leq |f(z_0)| \), by (18.3) we see that \( |f(z)| = |f(z_0)| \). It follows that \( |f(z)| \) is constant on the whole of \( U \), so that \( f(z) \) is constant.

*Second proof:* As \( f \) is holomorphic, we may find a power series representation of \( f \) about \( z_0 \),

\[
f(z) = \sum_{n} a_n(z - z_0)^n.
\]
Let $z = z_0 + re^{i\theta}$. Then
\[ f(z) = \sum_{n} a_n r^n e^{in\theta}. \]
\[ |f(z)|^2 = f(z)\bar{f}(z) = \sum_{m,n \in \mathbb{N}} \bar{a}_m a_n r^{m+n} e^{i(n-m)\theta}. \]

Hence
\[ \frac{1}{2\pi} \int_{\gamma} |f(z)|^2 \, dz = \frac{1}{2\pi} \sum_{m,n \in \mathbb{N}} \int_{0}^{2\pi} \bar{a}_m a_n r^{m+n} e^{i(n-m)\theta} \, d\theta \]
\[ = \sum_{n,m \in \mathbb{N}} |a_n|^2 r^{2n}. \]

But the integral on the LHS is no more than $|a_0|^2$, by (18.3). It follows that $a_n = 0$, for all $n \geq 1$. Thus $f$ is constant. \hfill \Box

**Lemma 18.3.** Let $\phi(x)$ be a continuous function such that $\phi(x) \leq k$. Then
\[ \frac{1}{b-a} \int_{a}^{b} \phi(x) \, dx \leq k, \]
with equality if and only if $\phi(x) = k$.

**Proof.** Easy. \hfill \Box

**Lemma 18.4.** Let $u$ be a non-constant harmonic function defined on a region $U$.

Then $u$ does not achieve its maximum on $U$.

**Proof.** Note that if $u$ is not constant then it is not locally constant. So it is enough to prove this locally. Thus we may assume that $u$ has a harmonic conjugate $v$, so that $u$ is the real part of a holomorphic function $f$. Then $e^f$ is holomorphic and $|e^f| = |e^u|$. \hfill \Box

**Lemma 18.5** (Schwarz’s Lemma). Let $f$ be a holomorphic function from the interior of the unit disc to the unit disc.

If $f(0) = 0$, then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. Furthermore if either $|f'(0)| = 1$ or $|f(z)| = |z|$ for some $z \neq 0$ then $f(z) = cz$, for some constant $c$, of absolute value one.

**Proof.** Note that
\[ g(z) = \begin{cases} \frac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0. \end{cases} \]
has a removable singularity at 0. Thus we may apply the maximum principle to $g(z)$.
On a circle of radius $r$, $|g(z)| \leq 1/r$. Thus $|g(z)| \leq 1/r$ on the circle $|z| \leq r$ by the maximum principle. Letting $r$ tend to one, we see that $|g(z)| \leq 1$. By the maximum principle $|g(z)| < 1$ for all $z$ unless it is constant and this gives the result. □