19. Residues

Let f be a holomorphic function with an isolated singularity at a. Pick a small circle γ centred at a and consider the integral

$$P = \int_\gamma f(z)\,\mathrm{d}z.$$

P is called a **period** of f. As the function $f(z) = \frac{1}{z-a}$ has period $2\pi i$ the function

$$g(z) = f(z) - \frac{R}{z-a}, \qquad \text{where} \qquad R = \frac{P}{2\pi i}$$

has period zero, with respect to γ. It follows that g is the derivative of some function.

Definition 19.1. *Let f be a holomorphic function with an isolated singularity at a. The **residue of** f **at** a is the unique complex number R, so that the function*

$$g(z) = f(z) - \frac{R}{z-a},$$

for some small $0 < |z-a| < \delta$, is the derivative of another function.

It is useful to employ the following notation for the residue,

$$R = \operatorname{Res}_{z=a} f(z).$$

Theorem 19.2 (Residue Theorem)**.** *Let U be a region and let f be a holomorphic function on $U - \{a_1, a_2, \dots\}$ with isolated singularities at $a_1, a_2, \dots$. Let γ be a path in U that does not contain any of the points $a_1, a_2, \dots$ and such that the winding number around any point outside U is zero.*

Then

$$\frac{1}{2\pi i}\int_\gamma f\,\mathrm{d}z = \sum_j n(\gamma; a_j)\operatorname{Res}_{z=a_j} f(z).$$

Proof. Pick small circles γ_j, centred at a_j, contained in U. Consider the path $\gamma' = \gamma - \sum n(\gamma; a_j)\gamma_j$. We want to apply Cauchy's integral formula to γ'. It suffices to check that the winding number of γ' about any complex number $a \in \mathbb{C} - (U - \{a_1, a_2, \dots, a_k\})$ is zero. Note that the regions of $\mathbb{C} - \gamma'$ are equal to the regions of $\mathbb{C} - \gamma$, union the small discs about each a_i. By assumption the only non-zero winding numbers for γ are about a_i. By definition of γ' the winding number of γ' about a_i is zero. It follows that γ' has zero winding number about any point in $a \in \mathbb{C} - (U - \{a_1, a_2, \dots, a_k\})$.

Thus by Cauchy's integral formula

$$\int_{\gamma'} f(z)\,\mathrm{d}z = 0.$$

Rearranging, we get

$$\int_{\gamma} f(z)\,\mathrm{d}z = \sum_j n(\gamma; a_j) P_j,$$

where

$$P_j = \int_{\gamma_j} f(z)\,\mathrm{d}z.$$

The result follows by definition of R_j. □

Of course, (19.2) is useless without an effective means of computing the residue:

Lemma 19.3. *Suppose that $f(z)$ has a pole of order one at a. Then*

$$\operatorname{Res}_{z=a} f(z) = \lim_{z\to a}(z-a)f(z).$$

Proof. By assumption

$$f(z) = \frac{b_{-1}}{z-a} + b_0 + b_1(z-a) + b_2(z-a)^2 + \cdots = \frac{b_{-1}}{z-a} + g(z),$$

where $g(z)$ is a holomorphic function. By definition the residue is b_{-1}. Clearly $b_{-1} = \lim_{z\to a}(z-a)f(z)$. □

One of the main uses of the residue Theorem is to compute contour integrals. For example, consider computing the following integral:

$$\int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x.$$

Consider the following contour. Let γ be the closed path, that starts at zero, goes along the real axis to R, describes a semi-circle of radius R and then traverses the x-axis from $-R$ to zero. Consider applying the Residue Theorem to

$$f(z) = \frac{1}{1+z^2}.$$

$f(z)$ has two isolated singularities at $z = \pm i$. The winding number of γ about the first is 1 and about the second is zero. The residue at $z = i$ can be computed in one of two ways.

For the first observe that

$$\frac{1}{1+z^2} = \frac{1}{(z-i)(z+i)} = \frac{-1}{2i(z+i)} + \frac{1}{2i(z-i)}.$$

Thus the residue at $z = i$ is by definition $1/2i$.

Alternatively multiply f by $(z-i)$, to get

$$\frac{1}{z+i}.$$

At $z = i$ we get $1/2i$.

Either way by the residue Theorem

$$\int_\gamma \frac{1}{1+z^2}\,\mathrm{d}z = \pi.$$

On the other hand the integral may by split into two parts. The integral along the real-axis from $-R$ to R and the integral along a semi-circle. Along the semi-circle,

$$|f(z)| \leq \frac{1}{R^2-1}$$

so that the integral along the semi-circle is at most

$$\pi\frac{R}{R^2-1}$$

which tends to zero as R tends to infinity.

As the function $\frac{1}{1+x^2}$ is even, it follows that the integral from $-R$ to R is twice the integral from 0 to R. Hence

$$\int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x = \frac{\pi}{2}.$$

Now consider the integral

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x.$$

Consider the integral

$$\int_\gamma \frac{e^{iz}}{z}\,\mathrm{d}z,$$

where γ is the contour that starts at ρ goes along the x-axis to R, goes around a semi-circle counterclockwise to $-R$, goes back to $-\rho$ and traverses a semi-circle, clockwise around the origin. The only pole of the function

$$f(z) = \frac{e^i z}{z},$$

is at the origin and the winding number of γ about the origin is zero. Thus by the residue Theorem, the integral of $f(z)$ around γ is zero. We split the integral into four pieces.

$$\int_\rho^R \frac{e^{ix}}{x}\,\mathrm{d}x + \int_{\gamma_0} f(z)\,\mathrm{d}z + \int_{-R}^{-\rho} \frac{e^{ix}}{x}\,\mathrm{d}x + \int_{\gamma_1} f(z)\,\mathrm{d}z.$$

The two integrals along the x-axis, when combined, give

$$\int_\rho^R \frac{e^{ix} - e^{-ix}}{x}\,\mathrm{d}x = 2i \int_\rho^R \frac{\sin x}{x}\,\mathrm{d}x.$$

Consider the behaviour around the big semi-circle.

$$\begin{aligned}
\left|\int_{\gamma_0} \frac{e^{iz}}{z}\,\mathrm{d}z\right| &= \left|\int_0^\pi e^{iRe^{i\theta}}\,\mathrm{d}\theta\right| \\
&\leq \int_0^\pi e^{-R\sin\theta}\,\mathrm{d}\theta \\
&\leq \int_0^\delta \mathrm{d}\theta + \int_\delta^{\pi-\delta} e^{-R\sin\delta}\,\mathrm{d}\theta + \int_{\pi-\delta}^\pi \mathrm{d}\theta \\
&\leq 2\delta + \pi e^{-R\sin\delta}.
\end{aligned}$$

As R tends to infinity, we may let δ approach zero. Thus the integral goes to zero.

Now consider the behaviour around the small semi-circle.

$$\int_\gamma \frac{e^{iz}}{z}\,\mathrm{d}z = \int_\gamma \frac{1}{z}\,\mathrm{d}z + \int_\gamma \frac{e^{iz} - 1}{z}\,\mathrm{d}z.$$

There are two ways to see that the first integral goes to zero as ρ goes to zero. Either use the Taylor series expansion of e^{iz}. Or use the fact that

$$\frac{e^{iz} - 1}{z}$$

is the derivative of a holomorphic function.

On the other hand, by direct computation, the first integral comes out as

$$\int_\gamma \frac{1}{z}\,\mathrm{d}z = \int_\pi^0 i\,\mathrm{d}\theta = -\pi i.$$

Thus, letting $R \to \infty$ and $\rho \to 0$, we get

$$2i \int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x - \pi i = 0,$$

so that

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}.$$

Finally consider

$$\int_0^\pi \log\sin\theta\,\mathrm{d}\theta.$$

Consider the function

$$1 - e^{2iz} = -2ie^{iz}\sin z.$$

As

$$1 - e^{2iz} = 1 - e^{-2y}\left(\cos 2x + i\sin 2x\right),$$

we see that this function takes on real negative values only if $y < 0$ and $x = n\pi/2$. So if we delete these half lines, we may assume that log is single-valued and holomorphic.

We now integrate $\log(1 - e^{2iz})$ along the rectangle with corners, 0, π, $\pi + iY$ and iY. At the points 0 and π we choose arcs of small quarter circles, of radius δ, to avoid these points.

By periodicity, the integrals along the vertical sides cancel. The integral along the top horizontal line goes to zero, as Y goes to infinity.

I claim that the same is true over the quarter circles. The imaginary part of the logarithm is bounded, so we only need worry about the real part. Now

$$\frac{|1 - e^{2iz}|}{|z|} \to 2,$$

for $z \to 0$ so that the logarithm behaves like $\log\delta$. As $\delta\log\delta$ tends to zero, the integral tends to zero around the first quarter circle. Similarly for the second quarter circle.

Thus

$$\int_0^\pi \log(-2ie^{ix}\sin x)\,\mathrm{d}x = 0.$$

Suppose we choose the standard branch of the logarithm. As x ranges between 0 and π we have

$$\log(e^{ix}) = ix \qquad \text{and} \qquad \log(-i) = -\pi i/2.$$

Thus

$$\pi\log 2 - \pi^2 i/2 + \int_0^\pi \log\sin x\,\mathrm{d}x + \pi^2/2i = 0.$$

and so

$$\int_0^\pi \log\sin x\,\mathrm{d}x = -\pi\log 2.$$