20. The argument principle

Theorem 20.1 (Argument Principle)**.** *Let $f(z)$ be a meromorphic function on a region U, with zeroes $a_1, a_2, \dots$ and poles $b_1, b_2, \dots$, repeated according to multiplicity.*

Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = \sum_j n(\gamma; a_j) - \sum_k n(\gamma; b_k),$$

where γ is any cycle that has zero winding number about any point of $\mathbb{C} - U$ and which does not contain of the points $a_1, a_2, \dots$ and $b_1, b_2, \dots$.

Proof. Suppose that $f(z) = (z-a)^h g(z)$, where $g(z)$ is holomorphic at a and $g(a) \neq 0$. Then

$$f'(z) = h(z-a)^{h-1}g(z) + (z-a)^h g'(z)$$

so that

$$\frac{f'(z)}{f(z)} = \frac{h}{z-a} + \frac{g'(z)}{g(z)}.$$

Thus the residue at a is h. The result now follows by the Residue Theorem. □

Note that the left hand side is the winding number of Γ about 0, the composition of Γ with f.

Theorem 20.2 (Rouché's Theorem)**.** *Let γ be a closed path in a region U such that the winding number of γ about any point of $\mathbb{C} - U$ is zero. Suppose that $n(\gamma; a)$ is zero 0 or 1 for any point a not on γ. Suppose that $f(z)$ and $g(z)$ are holomorphic on U and that*

$$|f(z) - g(z)| < |g(z)|,$$

on γ.

Then $f(z)$ and $g(z)$ have the same number of zeroes in the region enclosed by γ.

Proof. By the inequality, both f and g have no zeroes on γ. Moreover

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1$$

on γ. Thus $F(z) = f(z)/g(z)$ takes values in the disc with radius one and centre one. Thus if we apply (20.1) to $F(z)$, we see that $n(\Gamma; 0) = 0$, where Γ is the composition of γ and F. Hence $F(z)$ has the same number of zeroes and poles and this gives the result. □

We can generalise the argument principle to the case of

$$g(z)\frac{f'(z)}{f(z)},$$

where $g(z)$ is holomorphic on U and the zeroes or poles of f are disjoint from the zeroes of g. At a zero a of $f(z)$ of order h the residue is $hg(a)$ and at a pole of order h the residue is $-hg(a)$. Thus

$$\frac{1}{2\pi i}\int_{\gamma} g(z)\frac{f'(z)}{f(z)}\,\mathrm{d}z = \sum_{j} n(\gamma; a_j)g(a_j) - \sum_{k} n(\gamma; b_k)g(b_k).$$

Now consider the problem of constructing the inverse of a function. Suppose that $f(z)$ has local valency n, meaning

$$f(z) = w,$$

has n roots $z_j(w)$, for $0 < |w - w_0| < \epsilon$ and $|z - z_0| < \delta$. Applying the result above with $g(z) = z$, we get

$$\sum_{j=1}^{n} z_j(w) = \frac{1}{2\pi i}\int_{|z-z_0|=\epsilon} \frac{f'(z)}{f(z) - w} z\,\mathrm{d}z.$$

Thus if $n = 1$, the inverse function is explicitly

$$f^{-1}(w) = \frac{1}{2\pi i}\int_{|z-z_0|=\epsilon} \frac{f'(z)}{f(z) - w} z\,\mathrm{d}z.$$