21. More on Series

We want to investigate convergence of series. Typically we are given f_k on a region U_k and we want to find a limit f on the limit region U. Here $u \in U$ if and only if there is a k_0 so that $u \in U_k$, for all $k \geq k_0$. More often than not, the U_k are nested, so that

$$U_1 \subset U_2 \subset U_3 \ldots .$$

In this case $U = \bigcup U_k$.

Theorem 21.1 (Weierstrass' Theorem)**.** *Suppose that f_k is holomorphic on a region U_k and that f_k converges to f on U, uniformly on every compact subset of U.*

Then f is holomorphic on U and $f'_k(z)$ converges uniformly to $f'(z)$ on every compact set.

Proof. Pick $a \in U$ and pick $r > 0$ so that $|z - a| \leq r$ is a subset of U. As this disc is compact, it follows that there is a k_0 such that this disc is in U_k for all $k \geq k_0$. Let γ be any closed curve in the disc $|z - a| \leq r$. By Cauchy's Theorem,

$$\int_\gamma f_k(z)\, \mathrm{d}z = 0.$$

Taking limits, it follows that

$$\int_\gamma f(z)\, \mathrm{d}z = 0.$$

But then $f(z)$ is holomorphic by Morera's Theorem.

For the second statement, we start by giving another proof of the first statement. Cauchy's Integral Formula says

$$f_k(z) = \frac{1}{2\pi i} \int_C \frac{f_k(w)}{w - z}\, \mathrm{d}z,$$

where C is the circle $|w - z| \leq r$. Taking limits we get

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z}\, \mathrm{d}z,$$

and so $f(z)$ is holomorphic. The same proof applies to the derivative. Hence the second statement. □

One typical way to apply (21.1) is to the sequence of partial sums

$$f(z) = f_1(z) + f_2(z) + \cdots + f_k(z) + \ldots .$$

If this series converges uniformly on compact subsets, then $f(z)$ is holomorphic and we can find the derivative of f by taking the derivative of every term.

One must be careful about the behaviour at the boundary. For example, consider

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

This is uniformly convergent for $|z| \leq 1$. But the derivative

$$\sum_{n=0}^{\infty} \frac{z^n}{(n+1)}$$

is not convergent on the boundary. Indeed

$$f'(z) = \frac{\log(1-z)}{z}.$$

Theorem 21.2 (Hurwitz)**.** *Suppose that $f_k(z)$ are holomorphic and nowhere zero in a region U and that $f_k(z)$ converges to $f(z)$ uniformly on compact subsets.*

Then $f(z)$ is either identically zero or never zero.

Proof. Suppose that $f(z)$ is not the zero function. Then the zeroes of f are isolated. Then given any $a \in U$, there is an r such that $f(z)$ is not zero on the punctured neighbourhood $0 < |z-a| \leq r$ of a. Then $|f(z)|$ has a positive minimum on the circle C, $|z-a| = r$. Thus $1/f_n$ converges uniformly to $1/f$ on C. As $f_k'(z)$ converges uniformly to $f'(z)$ on C, we have

$$\lim_{n\to\infty} \frac{1}{2\pi i} \int_C \frac{f_n'(z)}{f_n(z)}\,\mathrm{d}z = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\,dz.$$

But the integrals on the LHS are all zero, since they give the number of zeroes of f_k in the circle. Thus the integal on the RHS is zero and f has no zeroes in C. □

Definition 21.3. *A series of the form*

$$\sum_{n=-\infty}^{n=\infty} a_n z^n,$$

*is called a **Laurent series**.*

Consider a series of the form

$$b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots.$$

Replacing z by $1/z$, it is clear that this series converges for $|z| > R$, for some R and convergence is uniform for $|z| > \rho$, where $\rho > R$.

Now a Laurent series is obviously the sum of an ordinary series in z and a series of the form above in z^{-1}. It follows that given a Laurent series, there are real numbers R_1 and R_2 such that the series converges in the annulus $R_1 < |z| < R_2$.

Theorem 21.4. *Let $f(z)$ be a function holomorphic in an annulus $R_1 < |z-a| < R_2$. Then we may express f as a Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n.$$

Proof. We may as well suppose that $a = 0$. It suffices to decompose f as

$$f(z) = f_1(z) + f_2(z)$$

where $f_1(z)$ is holomorphic for $|z| > R_1$ and $f_2(z)$ is holomorphic for $|z| < R_2$.

Define

$$\begin{aligned} f_1(z) &= -\frac{1}{2\pi i}\int_{|w|=r} \frac{f(w)\,\mathrm{d}w}{w-z} \qquad \text{for} \qquad R_1 < r < |z| \\ f_2(z) &= \frac{1}{2\pi i}\int_{|w|=r} \frac{f(w)\,\mathrm{d}w}{w-z} \qquad \text{for} \qquad |z| < r < R_2. \end{aligned}$$

Note that by Cauchy's Theorem two things follow:

(1) The integrals do not depend on r. Thus $f_1(z)$ is holomorphic for $|z| > R_1$ and $f_2(z)$ for $|z| < R_2$.
(2) $f(z) = f_1(z) + f_2(z)$, since the two circles have zero winding number about any point outside the annulus. □

It is interesting to use the proof of (21.4) to find the coefficients of the Laurent series. As we already know,

$$a_n = \frac{1}{2\pi i}\int_{|w-a|=r} \frac{f(w)\,\mathrm{d}w}{(w-a)^{n+1}},$$

for all $n \geq 0$.

It turns out that a similar expression holds for the negative coefficients. To prove this, it suffices to take the integral for $f_2(z)$ and transform it by the change of coordinates $w \longrightarrow w'$, where

$$w = a + \frac{1}{w'}.$$

We have

$$z = a + \frac{1}{z'} \qquad \text{and} \qquad \mathrm{d}w = -\frac{1}{w'^2}\mathrm{d}w',$$

also

$$\frac{f(w)}{w-z} = -w'z'\frac{f(a+1/w')}{w'-z'}$$

and

$$|w-a|=r \qquad \text{becomes} \qquad |w'|=\frac{1}{r}.$$

Thus

$$f_2(a+1/z')=\frac{1}{2\pi i}\int_{|w'|=1/r}\frac{z'}{w'}\frac{f(a+1/w')\,\mathrm{d}w'}{w'-z'}=\sum_{n=1}^{\infty} b_n z'^n.$$

As

$$\frac{z'}{w'}\frac{1}{w'-z'}=\frac{z'}{w'^2}\frac{1}{1-z'/w'}=\frac{z'}{w'^2}+\frac{z'^2}{w'^3}+\frac{z'^3}{w'^4}+\ldots,$$

we have

$$b_n=\frac{1}{2\pi i}\int_{|w'|=1/r}\frac{f(a+1/w')\,\mathrm{d}w'}{w'^{n+1}}=-\frac{1}{2\pi i}\int_{|w-a|=r} f(w)(w-a)^{n-1}\,\mathrm{d}w.$$

Thus the expression above for a_n, is valid for all n.