23. Infinite Products

Definition 23.1. *Let $p_1, p_2, \dots$ be an infinite sequence of non-zero complex numbers.*

$$p_1 p_2 p_3 \dots p_n \dots = \prod_{i=1}^{\infty} p_i$$

is defined to be the limit of the partial products

$$P_n = p_1 p_2 \dots p_n.$$

Note that if the product converges, then in fact $\lim_{n\to\infty} p_n = 1$. Indeed

$$p_n = \frac{P_n}{P_{n-1}}.$$

Given this, we set $p_n = 1 + a_n$ and rewrite our infinite product in the form,

$$\prod_{n=1}^{\infty}(1 + a_n).$$

If this product converges then $a_n \to 0$. Taking logarithms we obtain

$$\sum_{n=1}^{\infty} \log(1 + a_n).$$

Here we take the principal branch of the logarithm.

Theorem 23.2. *The infinite product*

$$\prod_{n=1}^{\infty}(1 + a_n)$$

converges if and only if

$$\sum_{n=1}^{\infty} \log(1 + a_n)$$

converges.

Proof. Suppose that the series converges. Let the partial sums be S_n and set

$$S = \lim_{n\to\infty} S_n.$$

Then $P_n = e^{S_n}$ and by continuity

$$P = \lim_{n\to\infty} P_n = \lim_{n\to\infty} e^{S_n} = e^S.$$

Thus one direction is clear. The only tricky part of the other direction is to deal with the fact that the logarithm has more than one branch. In general $\log P_n$ does not converge to $\log P$, but to one of its branches. Now as

$$\frac{P_n}{P} \to 1,$$

it follows that

$$\log\left(\frac{P_n}{P}\right) \to 0 \qquad \text{as } n \to \infty.$$

Now for every n there is an integer h_n such that

$$\log\left(\frac{P_n}{P}\right) = S_n - \log P + h_n(2\pi i).$$

Taking differences, we have

$$(h_{n+1} - h_n)(2\pi i) = \log\left(\frac{P_{n+1}}{P}\right) - \log\left(\frac{P_n}{P}\right) - \log(1 + a_n),$$

so that taking the argument

$$(h_{n+1} - h_n)(2\pi) = \arg\left(\frac{P_{n+1}}{P}\right) - \arg\left(\frac{P_n}{P}\right) - \arg(1 + a_n).$$

Now the first two terms on the right are approaching each other and the absolute value of the last term is at most π. Thus $h_{n+1} = h_n$ for n sufficiently large and so the series converges. □

Definition 23.3. *We say that the product P_n* ***converges absolutely*** *if the series*

$$\sum_{n=1}^{\infty} \log(1 + a_n)$$

converges absolutely.

Theorem 23.4. *The product*

$$\prod_{n=1}^{\infty} (1 + a_n)$$

converges absolutely if and only if

$$\sum_{n=1}^{\infty} a_n$$

converges absolutely.

Proof. It suffices to prove that

$$\sum_{n=1}^{\infty} |\log(1 + a_n)|$$

converges if and only if

$$\sum_{n=1}^{\infty} |a_n|$$

converges. As

$$\lim_{x \to 0} \frac{\log(1+x)}{x} = 1,$$

and as convergence of either series implies $a_n \to 0$, it follows that if $\epsilon > 0$ and n is sufficiently large then

$$(1-\epsilon)|a_n| < |\log(1+a_n)| < (1+\epsilon)|a_n|. \qquad \square$$