24. Representation of Functions as products

Lemma 24.1. *Let $f(z)$ be an entire function.*

Then $f(z)$ is never zero if and only if there is an entire function $g(z)$ such that

$$f(z) = e^{g(z)}.$$

Proof. One direction is clear. The function

$$e^{g(z)}$$

is never zero.

Otherwise consider

$$\frac{f'(z)}{f(z)}.$$

As this is entire, there is a holomorphic function $g(z)$ such that

$$g'(z) = \frac{f'(z)}{f(z)}.$$

Consider $f(z)e^{-g(z)}$. Its derivative is easily seen to be zero and the result follows easily. □

Lemma 24.2. *Let $f(z)$ be an entire function with a finite number of zeroes $a_1, a_2, \dots, a_k$ and a zero of order m at 0.*

Then there is an entire function $g(z)$ such that

$$f(z) = z^m e^{g(z)} \prod \left(1 - \frac{z}{a_n}\right).$$

Proof. Clear, since the ratio

$$\frac{f(z)}{z^m \prod \left(1 - \frac{z}{a_n}\right)},$$

is zero free. □

Again, we would like to do the same thing for a holomorphic function with an infinite number of zeroes. As before the only problem with the naive approach of extending the formula of (24.2) to the infinite case, are problems of convergence. In fact we are okay if

$$\sum \frac{1}{|a_k|}$$

converges and in this case we get absolute convergence on $|z| \leq R$. As before in general we need to modify the product to induce convergence.

So we try to introduce polynomials $p_n(z)$ such that

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{p_n(z)}$$

converges.

Taking logs this is equivalent to convergence of the series

$$\sum_{n=1}^{\infty} r_n(z),$$

where

$$r_n(z) = \log\left(1 - \frac{z}{a_n}\right) + p_n(z).$$

Here the branch of the logarithm should be chosen so that the argument of $r_n(z)$ lies between $-\pi$ and π (inclusive).

Now consider the Taylor series of

$$\log(1-y) = -y - y^2/2 - y^3/3 - \dots,$$

valid for $|y| < 1$. Of course, here we are using the standard branch of the logarithm. Ignoring the minus sign (which won't affect convergence), if we take $q_m(y)$ to be the first m terms, then what is left is

$$\frac{1}{m+1} y^{m+1} + \frac{1}{m+2} y^{m+2} + \dots \leq \frac{1}{m+1} y^{m+1} \left(1-y\right)^{-1},$$

valid for $|y| < 1$.

Fix R and discard any term such that $|a_n| \leq R$ (there are only finitely many such terms, so this will not affect convergence). If $|z| < R$ and

$$y = \frac{z}{a_n} \qquad \text{then} \qquad |y| = \frac{R}{|a_n|} < 1.$$

It follows that

$$|r_n(z)| \leq \frac{1}{m+1} \left(\frac{R}{|a_n|}\right)^{m+1} \left(1 - \frac{R}{|a_n|}\right)^{-1}.$$

Suppose that the series

$$\sum_{n=1}^{\infty} \frac{1}{m_n+1} \left(\frac{R}{|a_n|}\right)^{m_n+1},$$

converges, for some choice of m_n. Then we set $p_n(z) = q_{m_n}(z/a_n)$. Provided $|z|/|a_n| < 1$, then $r_n(z)$ is close to zero and its argument is then between $-\pi$ and π. Then $\sum r_n(z)$ is absolutely convergent and uniformly convergent for $|z| \leq R$.

Picking $m_n = n$, note that the series is absolutely convergent for all R, since the sum above is dominated by a geometric series with ratio less than one.

Theorem 24.3. *Let $a_1, a_2, \dots$ be a sequence of complex numbers that tends to infinity. Then there is an entire function with these zeroes.*

The most general function with these zeroes is of the form

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{q_{m_n}(\frac{z}{a_n})},$$

where m_n are integers, $g(z)$ is entire and

$$q_m(y) = y + (1/2)y^2 + (1/3)y^3 + \cdots + (1/m)y^n.$$

Corollary 24.4. *Every function meromorphic on the whole plane is the quotient of two entire functions.*

Proof. Let $F(z)$ be a function which is meromorphic on the whole plane. Pick an entire function $g(z)$ whose zeroes are located at the poles of $F(z)$. Then $f(z) = F(z)g(z)$ is an entire function and

$$F(z) = \frac{f(z)}{g(z)}.$$

□

The representation of (24.3) is considerably more interesting if we can choose $h = m_n$ to be constant. By the proof of the Theorem, this is equivalent to requiring convergence of the series

$$\sum_{n=1}^{\infty} \frac{\left(\frac{R}{|a_n|}\right)^{h+1}}{h+1}.$$

In other words we require convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{h+1}}.$$

Definition 24.5. *Pick the smallest value of h for which the series above converges. The corresponding product is called the* ***canonical product*** *and h is called the* ***genus*** *of the canonical product. Suppose that further $g(z)$ is a polynomial (for the canonical product). Then f is said to be of finite genus, and the* ***genus*** *of f is the larger of the degree of g and the genus of the canonical product.*

For example, an entire function of genus zero, is of the form

$$Cz^m \prod_{n=0}^{\infty} \left(1 - \frac{z}{a_n}\right)$$

with

$$\sum \frac{1}{|a_n|} < \infty.$$

Genus one is either of the form

$$Cz^m e^{\alpha z} \prod_{n=0}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{z/a_n},$$

with

$$\sum \frac{1}{|a_n|^2} < \infty, \qquad \text{and} \qquad \sum \frac{1}{|a_n|} = \infty,$$

or of the form

$$Cz^m e^{\alpha z} \prod_{n=0}^{\infty} \left(1 - \frac{z}{a_n}\right),$$

with

$$\sum \frac{1}{|a_n|} < \infty, \qquad \text{and} \qquad \alpha \neq 0.$$

Let us see what happens for $\sin \pi z$. The zeroes are located at the integers. As

$$\sum \frac{1}{n}$$

diverges and

$$\sum \frac{1}{n^2}$$

converges, we have a representation of the form

$$\sin \pi z = z e^{g(z)} \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{z/n}.$$

Taking the logarithmic derivative, we have

$$\cot \pi z = \frac{1}{z} + g'(z) + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right).$$

Here, we are allowed to differentiate term by term, as we have uniform convergence away from $z = n$. As we have already found such an expression for $\cot \pi z$, we see that $g'(z) = 0$ so that $g(z)$ is constant. Now

$$\lim_{z \to 0} \frac{\sin \pi z}{z} = \pi,$$

so that

$$e^{g(z)} = \pi.$$

It follows that

$$\sin \pi z = \pi z \prod_{n \neq 0} \left(1 - \frac{z}{n}\right) e^{z/n}.$$

Thus $\sin \pi z$ is an entire function of genus 1. Combining the $\pm n$ terms, we also have

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$